Gorenstein Projective Modules over Trivial Extension Rings

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Abstract

Let A be a ring and M an A-A-bimodule. Any left module over the trivial extension $A \ltimes M$ is corresponds to a pair (X, ϕ) where X is a left A-module and ϕ is an A-morphism from $M \otimes_A X$ to X such that $\phi(M \otimes_A \phi) = 0$. For any left $A \ltimes M$-module Y corresponding to a pair (X, ϕ), it will be shown that if Y is a Gorenstein projective $A \ltimes M$-module, then Cokϕ is a Gorenstein projective A-module and the sequence $M \otimes_A M \otimes_A X \xrightarrow{M \otimes_A \phi} M \otimes_A X \xrightarrow{\phi} X$ is exact under some assumptions on M.

Keywords: complete prpjective resolution, trivial extension, Gorenstein projective module

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1. Introduction

Throughout this paper, all rings are associative with the identity element, all modules are unital and left modules, unless otherwise stated.

Auslander and Bridger ([1]) introduced the G-dimension of finitely generated modules over noetherian rings and studied Gorenstein projective modules as modules of G-dimension zero. The definition of Gorenstein projective modules over arbitrary rings was given by Enochs and Jenda ([4]). It is noted that all modules over any quasi-Frobenius ring are Gorenstein projective and all Gorenstein projective modules over any ring of finite global dimension are projective. So it is natural to investigate Gorenstein projective modules over rings of infinite global dimension which is not quasi-Frobenius. There are, however, very few classes of rings whose Gorenstein projective modules have an explicit classification. For instance, Ringel ([8]) gave a classification of Gorenstein projective modules over Nakayama algebras, Kalck ([5]) gave that over gentle algebras and Chen, Shen and Zhou ([3]) gave that over monomial algebras. Zhang ([9]) described Gorenstein projective modules over triangulated algebras by using those over small algebras. Chen and Lu ([2]) generalized some results in [9] to those over tensor rings.

On the other hands, Mahdou and Ouarghi ([7]) studied Gorenstein projective modules over trivial extensions of commutative rings. Inspired by their works, Kimura investigated Gorenstein projective modules over trivial extensions of arbitrary ring

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in his master's thesis ([6]). Since triangulated algebras have a structure of trivial extensions, he generalized a result in [9] partially.

In this paper, we will refine the main theorem in [6] and will show a converce of the theorem under some assumptions.

2. Preliminaries

Let A be a ring. All modules will be left modules, unless otherwise stated. For A-A-bimodule M, the *trivial extension* $A \ltimes M$ of A by M is a ring whose group structure is that of $A \oplus M$, and whose multiplication is defined by

$$(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + m_1a_2)$$

for $(a_1, m_1), (a_2, m_2) \in A \oplus M$. The category Mod $A \ltimes M$ of $A \ltimes M$-modules has the following descriptions. Let $\mathcal{C}$ be the category whose objects are the pairs (X, ϕ) where X is an A-module and $\phi : M \otimes_A X \to X$ is an A-morphism such that $\phi(M \otimes_A \phi) = 0$, and morphisms between (X_1, ϕ_1) and (X_2, ϕ_2) are the A-morphisms $f : X_1 \to X_2$ such that $f\phi_1 = \phi_2\,(M \otimes_A f)$. The functor $F : \mathcal{C} \to$ Mod $A \ltimes M$ is defined as follows. For an object (X, ϕ) in $\mathcal{C}$, we define $F(X, \phi) = X$, where the $A \ltimes$ M-module structure on X is given by $(a, m)x = ax+\phi(m\otimes x)$ for $(a, m) \in A \ltimes$ M and $x \in X$. For a morphism $f : (X_1, \phi_1) \to (X_2, \phi_2)$, we define $F(f) = f$, where f is an $A \ltimes$ M-morphism via the $A \ltimes$ M-module structure on X_1 and X_2. Then the functor F is an equivalence. Throughout this paper, we shall identify these two categories.

Any $A \ltimes$ M-module (X, ϕ) induces the exact sequence

$$M \otimes_A \mathrm{Cok}\phi \to X \to \mathrm{Cok}\phi \to 0.$$

We denote by $\overline{\phi}$ the above A-morphism $M \otimes_A \mathrm{Cok}\phi \to X$. We can see that $\overline{\phi}$ is a monomorphism if and only if the sequence $M \otimes_A M \otimes_A X \xrightarrow{M \otimes_A \phi} M \otimes_A X \xrightarrow{\phi} X$ is exact. The regular $A \ltimes M$-module $A \ltimes M$ is the pair $(A \ltimes M, \phi)$ where $\phi(m_1 \otimes (a, m_2)) = (0, m_1a)$ for $m_1 \in M$ and $(a, m_2) \in A \ltimes M$. So we have that $\mathrm{Cok}\phi \xrightarrow{\sim} A$ and $\overline{\phi}$ is a monomorphism. Since any projective $A \ltimes M$-module (Q, ϕ) is a direct summand of a direct sum of copies of $A \ltimes M$, the cokernel of ϕ is a projective A-module and $\overline{\phi}$ is a monomorphism. So, any projective $A \ltimes M$-module is given by the pair $(P \oplus (M \otimes_A P), \phi)$ where P is a projective A-module and $\phi(m \otimes (x, y)) = (0, m \otimes x)$ for $m \in M$ and $(x, y) \in P \oplus (M \otimes_A P)$. We note that $\mathrm{Cok}\phi \xrightarrow{\sim} P$. For projective $A \ltimes M$-modules $Q_i = (P_i \oplus (M \otimes_A P_i), \phi_i)$ $(i = 1, 2)$, any $A \ltimes M$-morphism $f : Q_1 \to Q_2$ is given by A-morphisms $g : P_1 \to P_2$ and $h : P_1 \to M \otimes_A P_2$ such that $f(x, m \otimes y) = (g(x), h(x) + m \otimes g(y))$ for $x \in P_1$ and $m \otimes y \in M \otimes_A P_1$ since $f\phi_1 = \phi_2(M \otimes_A f)$. This correspondence gives the isomorphism between $\mathrm{Hom}_{A\ltimes M}(Q_1, Q_2)$ and $\mathrm{Hom}_A(P_1, P_2) \oplus \mathrm{Hom}_A(P_1, M \otimes_A P_2)$ as groups. For (g_i, h_i) in $\mathrm{Hom}_A(P_i, P_{i+1}) \oplus \mathrm{Hom}_A(P_i, M \otimes_A P_{i+1})$ $(i = 1, 2)$, the composition of (g_1, h_1) and (g_2, h_2) corresponds to $(g_2g_1, h_2g_1 + (M \otimes_A g_2)h_1)$ in $\mathrm{Hom}_A(P_1, P_3) \oplus \mathrm{Hom}_A(P_1, M \otimes_A P_3)$.

3. Complete projective resolutions

A *complete projective resolution* $P^\bullet : \cdots \to P^i \xrightarrow{d^i} P^{i+1} \to \cdots$ is an exact sequence of projective A-modules, such that $\mathrm{Hom}_A(P^\bullet, P)$ is exact for any projective A-module P. An A-module X is *Gorenstein-projective* if there is a complete projective resolution $P^\bullet$ with $\mathrm{Im}\, d^0 \cong X$.

Let $Q^\bullet : \cdots \to Q^i \xrightarrow{\partial^i} Q^{i+1} \to \cdots$ be a complex of projective $A \ltimes M$-modules with $Q^i = (P^i \oplus (M \otimes_A P^i), \phi^i)$ and $\partial^i(x, m\otimes y) = (d^i(x), h^i(x)+m\otimes d^i(y))$ where $d^i : P^i \to P^{i+1}$ and $h^i : P^i \to M \otimes_A P^{i+1}$ are A-morphisms. Then $P^\bullet : \cdots \to P^i \xrightarrow{d^i} P^{i+1} \to \cdots$ is a complex of projective A-modules and $h^\bullet = (h^i)_{i\in\mathbb{Z}}$ is a morphism $P^\bullet \to (M \otimes_A P^\bullet)[1]$ of complexes, where $(M \otimes_A P^\bullet)[1]$ is the 1-shifted complex of $M \otimes_A P^\bullet$. We note that the mapping cone of $h^\bullet$ is the complex $Q^\bullet$.

The complex $\mathrm{Hom}_{A\ltimes M}(Q^\bullet, Q)$ for a projective $A\ltimes M$-module $Q = (P \oplus (M \otimes_A P), \phi)$ has the following description. The i-th component $\mathrm{Hom}_{A\ltimes M}(Q^i, Q)$ is isomorphic to $\mathrm{Hom}_A(P^i, P) \oplus \mathrm{Hom}_A(P^i, M \otimes_A P)$. The morphism $\mathrm{Hom}_{A\ltimes M}(\partial^i, Q) : \mathrm{Hom}_{A\ltimes M}(Q^{i+1}, Q) \to \mathrm{Hom}_{A\ltimes M}(Q^i, Q)$ is isomorphic to

$$\begin{array}{ccc} \mathrm{Hom}_A(P^{i+1}, P) & \xrightarrow{\mathrm{Hom}_A(d^i, P)} & \mathrm{Hom}_A(P^i, P) \\ \oplus & \searrow^{h_i^P} & \oplus \\ \mathrm{Hom}_A(P^{i+1}, M \otimes_A P) & \xrightarrow[\mathrm{Hom}_A(d^i, M \otimes_A P)]{} & \mathrm{Hom}_A(P^i, M \otimes_A P) \end{array}$$

where $h_i^P(f) = (M \otimes_A f)h^i$ for $f \in \mathrm{Hom}_A(P^{i+1}, P)$. Then $h_\bullet^P = (h_i^P)_{i\in\mathbb{Z}}$ is a morphism $\mathrm{Hom}_A(P^\bullet, P) \to \mathrm{Hom}_A(P^\bullet, M \otimes_A P)[-1]$ of complexes. We note that the mapping cone of $h_\bullet^P$ is the complex $\mathrm{Hom}_{A \ltimes M}(Q^\bullet, Q)$. Hence $Q^\bullet$ is a complete projective resolution if and only if $h^\bullet$ and $h_\bullet^P$ are quasi-isomorphisms for any projective A-module P.

Proposition 3.1. *Let A be a ring and M an A-A-bimodule. For a complex of projective A-modules $P^\bullet : \cdots \to \mathrm{P}^i \xrightarrow{d^i} \mathrm{P}^{i+1} \to \cdots$ and a morphism of complexes $h^\bullet : P^\bullet \to (M \otimes_A P^\bullet)[1]$, we denote the mapping cone of $h^\bullet$ by $Q^\bullet : \cdots \to Q^i \xrightarrow{\partial^i} Q^{i+1} \to \cdots$.*

(1) *Assume that $Q^\bullet$ is a complete projective resolution as $A \ltimes M$-modules. Then $P^\bullet$ is a complete projective resolution if and only if $M \otimes_A P^\bullet$ and $\mathrm{Hom}_A(P^\bullet, M \otimes_A P)$ are exact for any projective A-module P.*

(2) *Assume that $P^\bullet$ is a complete projective resolution. Then $Q^\bullet$ is a complete projective resolution as $A \ltimes M$-modules if and only if $M \otimes_A P^\bullet$ and $\mathrm{Hom}_A(P^\bullet, M \otimes_A P)$ are exact for any projective A-module P.*

(3) *Assume that $Q^\bullet$ is exact. Let (X^i, ϕ^i) be the $A \ltimes M$-module $\mathrm{Im}\partial^i$. Then $P^\bullet$ is exact if and only if $\overline{\phi}^i$ is a monomorphism for any integer i.*

Proof. (1) Since $Q^\bullet$ is a complete projective resolution, $h^\bullet$ and $h_\bullet^P$ are quasi-isomorphisms for any projective A-module P. Hence $P^\bullet$ is exact if and only if $M \otimes_A P^\bullet$ is exact, and $\mathrm{Hom}_A(P^\bullet, P)$ is exact for any projective A-module P if and ony if $\mathrm{Hom}_A(P^\bullet, M \otimes_A P)$ is exact for any projective A-module P.

(2) Since $P^\bullet$ is a complete projective resolution, $P^\bullet$ and $\mathrm{Hom}_A(P^\bullet, P)$ are exact for any projective A-module P. Hence $h^\bullet$ and $h_\bullet^P$ are quasi-isomorphisms for any projective A-module P if and only if $M \otimes_A P^\bullet$ and $\mathrm{Hom}_A(P^\bullet, M \otimes_A P)$ are exact for any projective A-module P.

(3) Since $Q^\bullet$ is exact, we heve the commutative diagram

$$\begin{array}{ccccccc} M \otimes_A Q^{i-1} & \longrightarrow & Q^{i-1} & \longrightarrow & P^{i-1} & \longrightarrow & 0 \\ \downarrow{\scriptstyle M \otimes_A \partial^{i-1}} & & \downarrow{\scriptstyle \partial^{i-1}} & & \downarrow{\scriptstyle d^{i-1}} & & \\ M \otimes_A Q^i & \longrightarrow & Q^i & \longrightarrow & P^i & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ M \otimes_A X^i & \xrightarrow{\phi^i} & X^i & \longrightarrow & \mathrm{Cok}\phi^i & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array}$$

with exact rows and columns. Thus the commutative diagram

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M \otimes_A P^{i-2} & \longrightarrow & Q^{i-1} & \longrightarrow & P^{i-2} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle M \otimes_A d^{i-2}} & & \downarrow{\scriptstyle \partial^{i-2}} & & \downarrow{\scriptstyle d^{i-2}} & & \\ 0 & \longrightarrow & M \otimes_A P^{i-1} & \longrightarrow & Q^{i-1} & \longrightarrow & P^{i-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle M \otimes_A d^{i-1}} & & \downarrow{\scriptstyle \partial^{i-1}} & & \downarrow{\scriptstyle d^{i-1}} & & \\ 0 & \longrightarrow & M \otimes_A P^i & \longrightarrow & Q^i & \longrightarrow & P^i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with exact rows induces the exact sequence

$$\mathrm{H}^{i-1}(Q^\bullet) \longrightarrow \mathrm{H}^{i-1}(P^\bullet) \longrightarrow M \otimes_A \mathrm{Cok}\phi^i \xrightarrow{\overline{\phi}^i} \mathrm{X}^i \longrightarrow \mathrm{Cok}\phi^i \longrightarrow 0.$$

Hence $P^\bullet$ is exact if and only if $\overline{\phi}^i$ is a monomorphism for any integer i because $Q^\bullet$ is exact.

□

The following theorem was proved by Kimura in [6]. We will give the proof of the theorem for the convenience of readers.

Theorem 3.2 ([6]). *Let A be a ring, M an A-A-bimodule and (X,ϕ) an $A \ltimes M$-module. Assume that $M \otimes_A P^\bullet$ and $\mathrm{Hom}_A(P^\bullet, M \otimes_A P)$ are exact for any complete projective resolution $P^\bullet$ and any projective A-module P. If $\mathrm{Cok}\phi$ is a Gorenstein-projective A-module and $\overline{\phi}$ is a monomorphism, then (X, ϕ) is a Gorenstein-projective $A \ltimes M$-module. Especially, for any Gorenstein-projective A-module Y, $(A \ltimes M) \otimes_A Y$ is a Gorenstein-projective $A \ltimes M$-module.*

Proof. Let $P^\bullet : \ \cdots \to P^i \xrightarrow{d^i} P^{i+1} \to \ \cdots$ be a complete projective resolution with $\mathrm{Im}\, d^0$ isomorphic to $\mathrm{Cok}\phi$. Then d^0 is the composition of an epimorphism $\alpha : P^0 \to \mathrm{Cok}\phi$ and a monomorphism $\beta : \mathrm{Cok}\phi \to P^1$. Since $\overline{\phi}$ is monomorphism, we have the exact sequence $0 \to M \otimes_A \mathrm{Cok}\phi \xrightarrow{\bar\phi} X \to \mathrm{Cok}\phi \ \to 0$. Since P^0 is projective, α factors through the epimorphism $X \to \mathrm{Cok}\phi$. Since $\mathrm{Hom}_A(P^\bullet, M \otimes_A P)$ is exact for any projective A-module P, $\mathrm{Ext}^1_A(\mathrm{Cok}\phi, M \otimes_A P) = 0$. Thus $M \otimes_A \beta$ factors through $\overline{\phi}$. Therefore we obtain the commutative diagram

$$\begin{array}{ccccccccc}
 & & P^{-1} & \xrightarrow{d^{-1}} & P^0 & \xrightarrow{\alpha} & \mathrm{Cok}\phi & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M \otimes_A \mathrm{Cok}\phi & \xrightarrow{\bar\phi} & X & \longrightarrow & \mathrm{Cok}\phi & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M \otimes_A \mathrm{Cok}\phi & \xrightarrow{M \otimes_A \beta} & M \otimes_A P^1 & \xrightarrow{M \otimes_A d^1} & M \otimes_A P^2 & &
\end{array}$$

with the exact rows. We denote by h^0 the compsition map of $P^0 \to X$ and $X \to M \otimes_A P^1$ in the above diagram. Then we have $(M \otimes_A d^1)h^0 d^1 = 0$. Assume that there exists a morphism $h^i : P^i \to M \otimes_A P^{i+1}$ with $(M \otimes_A d^{i+1})h^i d^{i+1} = 0$. Since P^{i-1} is projective and $M \otimes_A P^\bullet$ is exact, there exists a morphism $h^{i-1} : P^{i-1} \to M \otimes_A P^i$ such that $(M \otimes_A d^i)h^{i-1} + h^i d^{i-1} = 0$. Thus $(M \otimes_A d^i)h^{i-1}d^i = 0$. Since $\mathrm{Hom}_A(P^\bullet, M \otimes_A P)$ is exact for any projective A-module P, there exists a morphism $h^{i+1} : P^{i+1} \to M \otimes_A P^{i+2}$ such that $(M\otimes_A d^{i+2})h^{i+1} + h^i d^{i+1} = 0$. Thus $(M \otimes_A d^{i+2})h^{i+1}d^{i+2} = 0$. Hence we obtain a morphism of complex $h^\bullet : P^\bullet \to (M \otimes_A P)[1]$. We denote the mapping cone of $h^\bullet$ by $Q^\bullet : \cdots \to Q^i \xrightarrow{\partial^i} Q^{i+1} \to \cdots$. Then , by Proposition 3.1, $Q^\bullet$ is a complete projective resolution. It is not hard to see that $\mathrm{Im}\, \partial^0 \cong (X, \phi)$ as $A \ltimes M$-modules. Hence (X, ϕ) is a Gorenstein-projective $A \ltimes M$-module. □

For an A-A-bimodule M, we write $M^{\otimes 0} = A$ and $M^{\otimes j} = M \otimes_A M^{\otimes (j-1)}$ for any positive integer j. The converse of the above theorem will be given under some assumptions.

Theorem 3.3. *Let A be a ring, M an A-A-bimodule and (X, ϕ) an $A \ltimes M$-module. Assume that $M^{\otimes n} = 0$ for some n, $\mathrm{Tor}^A_i(M^{\otimes j}, M) = 0$ for any positive integer i and j, the flat dimension of the right A-module M is finite and the injective dimension of the left A-module $M \otimes_A P$ is finite for any projective A-module P. If (X, ϕ) is a Gorenstein-projective $A \ltimes M$-module, then $\mathrm{Cok}\phi$ is a Gorenstein-projective A-module and $\overline{\phi}$ is a monomorphism.*

Proof. Let $Q^\bullet : \cdots \to Q^i \xrightarrow{\partial^i} Q^{i+1} \to \cdots$ be a complete projective rosolution of $A \ltimes M$-modules with $\mathrm{Im}\, \partial^0 \cong (X, \phi)$. For any integer i, $Q^i = P^i \oplus (M\otimes_A P^i)$ as A-modules for some projective A-module P^i. Let $P^\bullet : \cdots \to P^i \xrightarrow{d^i} P^{i+1} \to \cdots$ be the induced complex. There exists a morphism of complex $h^\bullet : P^\bullet \to (M\otimes_A P^\bullet)[1]$ such that the mapping cone of $h^\bullet$ is $Q^\bullet$. Thus the mapping cone of $M^{\otimes j} \otimes_A h^\bullet$ is $M^{\otimes j} \otimes_A Q^\bullet$ for any non-negative integer j. Since $\mathrm{Tor}^A_i(M^{\otimes j}, M) = 0$ for any positive integer i and j, $\mathrm{Tor}^A_i(M^{\otimes j}, Q^k) = \mathrm{Tor}^A_i(M^{\otimes j}, P^k) \oplus \mathrm{Tor}^A_i(M^{\otimes j}, M \otimes_A P^k) = 0$ for any integer k. Therefore $\mathrm{Tor}^A_i(M^{\otimes j}, \mathrm{Im}\, \partial^k) = \mathrm{Tor}^A_{i+1}(M^{\otimes j}, \mathrm{Im}\, \partial^{k+1})$ for any positive integer i and j. Since the flat dimension of the right A-module M is finite and $\mathrm{Tor}^A_i(M^{\otimes j}, M) = 0$ for any positive integer i and j, the flat dimension of the right A-module $M^{\otimes j}$ is finite for any non-negative integer j and so, $\mathrm{Tor}^A_i(M^{\otimes j}, \mathrm{Im}\, \partial^k) = 0$. Hence $M^{\otimes j} \otimes_A Q^\bullet$ is exact, namely, $M^{\otimes j} \otimes_A h^\bullet : M^{\otimes j} \otimes_A P^\bullet \to (M^{\otimes (j+1)} \otimes_A P^\bullet)[1]$ is a quasi-isomorphism for any non-negative integer j. Since $M^{\otimes n} = 0$ for some n, $M^{\otimes n} \otimes_A P^\bullet$ is exact, so we obtain that $M^{\otimes j} \otimes_A P^\bullet$ is exact for any non-negative integer j. Since the injective dimension of $M \otimes_A P$ is finite for any projective A-module P, $\mathrm{Hom}_A(P^\bullet, M \otimes_A P)$ is exact. Therefore, by Proposition 3.1, $P^\bullet$ is a complete projective resolution. It is not hard to see that $\mathrm{Im} d^0 \cong \mathrm{Cok}\phi$ as A-modules. Hence $\mathrm{Cok}\phi$ is a

Gorenstein projective A-module. Moreover, by Proposition 3.1, $\overline{\phi}$ is a monomorphism. □

Corollary 3.4. *Let A be a ring, M an A-A-bimodule and* (X, ϕ) *an* $A \ltimes M$*-module. Assume that* $M^{\otimes n} = 0$ *for some n,* $\mathrm{Tor}_i^A(M^{\otimes j}, M) = 0$ *for any positive integer i and j, the flat dimension of the right A-module M is finite and the injective dimension of the left A-module* $M \otimes_A P$ *is finite for any projective A-module P. Then* (X, ϕ) *is a Gorenstein projective* $A \ltimes M$*-module if and only if* $\mathrm{Cok}\phi$ *is a Gorenstein projective A-module and* $\overline{\phi}$ *is a monomorphism.*

Proof. By Theorem 3.3, if (X, ϕ) is a Gorenstein projective $A \ltimes M$-module, then $\mathrm{Cok}\phi$ is a Gorenstein projective A-module and $\overline{\phi}$ is a monomorphism. Since the flat dimension of the right A-module M is finite, $M \otimes_A P^{\bullet}$ is exact for any complete projective resolution $P^{\bullet}$. Since the injective dimension of the left A-module $M \otimes_A P$ is finite for any projective A-module P, $\mathrm{Hom}_A(P^{\bullet}, M \otimes_A P)$ is exact for any complete projective resolution $P^{\bullet}$ and any projective A-module P. So, by Theorem 3.2, if $\mathrm{Cok}\phi$ is a Gorenstein projective A-module and $\overline{\phi}$ is a monomorphism, then (X, ϕ) is a Gorenstein projective $A \ltimes M$-module. □

Example 3.5. Let A be a finite dimensional algebra over a field k and E an injective A-module. Assume that there exists an idempotent e in A such that $eE = 0$, for instance, any hereditary algebra A which is not simple has such an injective module and an idempotent. Then the A-A-bimodule $M = E \otimes_k eA$ fulfills the conditions in Corollary 3.4.

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自明拡大環上のゴレンシュタイン射影加群

長　瀬　　　潤*

数学分野

要　旨

環 A に対し，M を A-A両側加群とする。自明拡大環 $A \ltimes M$ 上の任意の左加群は，左 A 加群 X と $M \otimes_A X$ から X への A 準同型写像 ϕ の組 (X, ϕ) で $\phi(M \otimes_A \phi) = 0$ を満たすものと対応している。この論文では，M がある条件を満たしているとき，組 (X, ϕ) に対応する $A \ltimes M$ 加群 Y に対し，Y がゴレンシュタイン射影加群であれば，Cokϕ がゴレンシュタイン射影 A加群であり，列 $M \otimes_A M \otimes_A X \xrightarrow{M \otimes_A \phi} M \otimes_A X \xrightarrow{\phi} X$ が完全列となることが示される。

キーワード：完全射影分解，自明拡大環，ゴレンシュタイン射影加群

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