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Abstract

Convex ideals in a partially ordered ring give a naturally induced partial order in their residue class rings ([1]). The similar holds for convex subgroups in a partially ordered group. The partial orders in rings or groups are respectively determined by semi-cones ([3, 4]) or positive subsets ([7]). In this paper, we give a characterization for convexity of subgroups in the direct product groups with some canonical positive subsets. Also, we give a method of the construction of the product extension rings which have semi-cones with a similar type of those positive sets, and we consider convexity of ideals there.

Keywords: direct product group, product extension ring, partial order, positive set, semi-cone, convex subgroup, convex ideal, group monomorphism

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1. Introduction

The symbol G means a non-zero additive group (abbreviated *group*). The symbol R means a non-zero commutative ring with the identity 1.

The symbol $\mathbb{Z}$ is the ring of integers, and $\mathbb{Z}^*$ (resp. $\mathbb{N}$) is the set of non-negative (resp. positive) integers.

As is well-known, G (resp. R) is a *partially ordered group* (resp. *partially ordered ring*) if it has a partial order $\leq$ satisfying (i) (resp. (i) and (ii)) below.

(i) $a \leq b$ implies $a + x \leq b + x$ for all x.

(ii) $a \leq b$ and $0 \leq x$ imply $ax \leq bx$.

Let us recall that a partial order in G satisfying (i) is determined by a *positive subset* P of G ([7]); that is, $P + P \subset P$ and $P \cap -P = 0$, here $P + P = \{x + y \mid x, y \in P\}$, $-P = \{-x \in G \mid x \in P\}$. Namely, for a positive subset P of G, define $x \leq_P y$ by $y - x \in P$, then $\leq_P$ is a partial order satisfying (i) in G. Conversely, for a partial order $\leq$ satisfying (i) in G, $P = \{x \in G \mid x \geq 0\}$

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is a positive subset of G with $\leq = \leq_P$. For a positive subset P in G, $-P$ is also a positive subset. A subset S of R is a *non-negative semi-cone* ([3]) (abbreviated *semi-cone* ([4])), if S is a positive subset satisfying $SS \subset S$, here $SS = \{xy \mid x, y \in S\}$. A semi-cone S is a *non-negative cone* ([2]) (abbreviated *cone* ([6])), if $R = S \cup -S$. A partial order (resp. order) in R satisfying (i) and (ii) is determined by a semi-cone (resp. cone), and then a ring with a semi-cone (resp. cone) is precisely a partial ordered ring (resp. ordered ring). (The concepts of semi-cones, cones, etc. are classical or well-known).

Let H be a subgroup of G. For a positive subset P of G (i.e., G has a partial order $\leq = \leq_P$), H is *convex for P* (or *P-convex*) if whenever $z \leq x \leq y$ and $z, y \in H$, then $x \in H$, here we can assume $z = 0 \leq x \leq y \in H \cap P$. The similar is true of a subgroup of the direct product group $G \times G$ with a positive subset. For positive subsets P and T of G with $P \subset T$, if H is T-convex, H is P-convex.

For a (proper) subgroup H and a positive subset P of G, H is convex for P iff the residue class group G/H has a positive subset $\varphi(P)$ by the natural map φ. For a (proper) ideal I and a semi-cone S of R, the similar holds for the residue class ring R/I (see [1]). We consider the ordered ring (resp. partially ordered ring) R/I in terms of a cone (resp. semi-cone) S of R in [2, 3], etc.

For $a, b \in R$, let $(R \ltimes R; a, b)$ be a ring $(R \times R, +, *)$ defined by the addition $+$ and multiplication $*$ below, and we call $(R \ltimes R; a, b)$ the *product extension ring* of R ([5]): For $(x_1, y_1), (x_2, y_2) \in R \times R$, let

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$
$$(x_1, y_1) * (x_2, y_2) = (x_1x_2 + ay_1y_2, x_1y_2 + y_1x_2 + by_1y_2).$$

The ring $(R \ltimes R; a, b)$ is a commutative R-algebra which contains a subring isomorphic to R, and it gives useful ring-theoretic constructions or examples. The direct product ring $R \times R$ is not an integral domain. On the other hand, the ring $(R \ltimes R; a, b)$ is possibly an integral domain or a field (if so is R), specially, for the real number field $\mathbb{R}$, $(\mathbb{R} \ltimes \mathbb{R}; -1, 0)$ is a field isomorphic to the complex number field (for these, see [5]).

Throughout this paper, *the symbol P means a positive subset of G with* $P \neq 0$, *and let* $P_0 = \{x \in P \mid x \neq 0\}$. *The symbol S means a semi-cone of R with* $S \neq 0$, *and let* $S_0 = \{x \in S \mid x \neq 0\}$.

The symbol (G, P) means that G is a partially ordered group with the partial order $\leq = \leq_P$, and the similar is true of the symbol (G', P'), etc.

For P of G, let us recall the following canonical positive subsets of $G \times G$ which are induced by P ([7]).

$D_0 = \{(x, y) \in P \times P \mid x = y \in P\}$.
$D_1 = \{(x, y) \in P \times P \mid x - y \in P\}$.
$D_2 = \{(x, y) \in P \times P \mid y - x \in P\}$.
$L_0 = P \times P$.
$L = L_0 \cup (P_0 \times G)$. (*Lexicographic set*)

Throughout this paper, let us use the symbol D_3 instead of L_0 (i.e., $D_3 = L_0$). We use the symbol D_i instead of "D_i $(i = 0, 1, 2, 3)$".

Clearly, $D_0 = D_1 \cap D_2$, $D_1 \cup D_2 \subset D_3 \subset L$. For D_i and L induced by a semi-cone S of R (instead of P of G), D_i are semi-cones in the direct product ring $R \times R$, but L is never a semi-cone there. On the other hand, these D_i or L need not be semi-cones in the product extension ring $(R \ltimes R; a, b)$ (see [6]).

In [6], for L we give a characterization for ideals in the product extension rings to be convex, assuming L is a semi-cone.

Analogously, we give a characterization for subgroups in the (direct) product groups to be convex for (the positive subset) L ([7]).

In this paper, we give a characterization for convexity of subgroups in $G \times G$ with the positive subsets D_i induced by P under (G, P) being Archimedean. We apply it to ideals in $R \times R$ with the semi-cones D_i induced by S, and ideals in $(R \ltimes R; a, b)$ with assuming D_i are semi-cones. To avoid this assumption, for $(R \ltimes R$; a, b) and the positive subsets D_i, we systematically construct a product extension ring $(R' \ltimes R'; a', b')$ satisfying the following: (i) it contains $(R \ltimes R; a, b)$ as a group $R \times R$ with D_i, (ii) it has semi-cones D_i' of a similar type of D_i with $D_i' * D_i' = 0$, and (iii) for an ideal I of $(R \ltimes R; a, b)$, there exists an ideal I' of $(R' \ltimes R'; a', b')$ such that I is convex for D_i as the group $R \times R$ iff so is I' for D_i'.

2. Group monomorphisms and convexity

The following is a basic proposition on preservation of convexity.

Proposition 2.1. *Let $h : G \to G'$ be a group monomorphism. Let T be a positive subset of G, and let $T' = h(T)$. Then the following hold.*

(1) *T' is a positive subset of G'.*

(2) *For a subgroup H (resp. H') of G (resp. G'), suppose (T) $h(H \cap T) = H' \cap T'$ holds. Then H is convex for T iff H' is convex for T'.*

Proof. (1) is obvious. For (2), the if part is routinely shown by (T), noting $0 \leq x \leq y$, $\leq = \leq_T$ implies $0 \leq h(x) \leq h(y)$, $\leq = \leq_{T'}$. To see the only if part, let $0 \leq x' \leq y' \in H' \cap T'$, $\leq = \leq_{T'}$. Then $y' = h(y)$ for some $y \in H \cap T$ by (T), and $x' = h(x)$ for some $x \in T$. But $y' - x' = h(y - x) \in T'$. Then $0 \leq x \leq y \in H \cap T$, $\leq = \leq_T$. Thus, $x \in H \cap T$ by convexity of H for T. Hence $x' = h(x) \in H'$ by (T). □

Remark 2.2. In Proposition 2.1(2), (T) is essential even if h is the identity map (putting $G = G' = \mathbb{Z}$, $T = T' = 2\mathbb{Z}^*$, and $H = 2\mathbb{Z}$, $H' = 4\mathbb{Z}$ and vice versa.

Generally, the following holds for convexity of subgroups of $\mathbb{Z}$ (cf. [3]).

(Proposition) *For a positive subset T, and a non-zero subgroup H of $\mathbb{Z}$, H is convex for T iff $T \subset H$* (indeed, the if part is obvious. For the only if part, we can put $H = m\mathbb{Z}$ for some $m \in \mathbb{N}$. Let $n \in T$. Then $0 \leq n \leq mn \in H$, here $\leq = \leq_T$. Thus $n \in H$ by the convexity of H. Hence $T \subset H$).

Let $p_1, p_2 : G \times G \to G$ be the projections defined by $p_1(x, y) = x$, $p_2(x, y) = y$.

Let H be a subgroup of $G \times G$, and T be a positive subset of $G \times G$ with $T \subset P \times P$. Related to convexity of H, let us recall the following conditions (p_i) for T ([7, 9]).

(p_1) $0 \leq x \leq y \in p_1(H \cap T)$ implies $(x, 0) \in H$.

(p_2) $0 \leq x \leq y \in p_2(H \cap T)$ implies $(0, x) \in H$.

For a subgroup H', and a positive subset T' of $G' \times G'$ with $T' \subset P' \times P'$, similarly define conditions (p'_i) for T' as (p_i) by the projections $p'_i : G' \times G' \to G'$.

Remark 2.3. Let H be a subgroup of $G \times G$. Then the following hold.

(1) For the positive subsets D_i of $G \times G$, if (p_1) and (p_2) hold, then H is convex. For D_3, the converse holds (but for the other D_i, the converse need not hold).

(2) For D_i $(i = 0, 1, 2)$, if H is convex, then $(p_1) \Leftrightarrow (p_2)$ holds.

Indeed, (1) is shown in [7], but for the parenthetic part, consider a subgroup $H = \{(x, x) \mid x \in \mathbb{Z}\}$ of $\mathbb{Z} \times \mathbb{Z}$. In (2), for D_1, to see $(p_1) \Rightarrow (p_2)$, let $0 \le x \le y \in p_2(H \cap D_1)$. Take $(x', y) \in H \cap D_1$. Then $0 \le x \le x' \in p_1(H \cap D_1)$, and $(0, 0) \le (x, x) \le (x', y) \in H \cap D_1$. Thus $(x, 0)$ and $(x, x) \in H$ by the assumption, hence $(0, x) \in H$. Thus (p_2) holds. Similarly, for D_0, $(p_1) \Leftrightarrow (p_2)$ holds, and for D_2, $(p_2) \Rightarrow (p_1)$ holds. For D_1, to see $(p_2) \Rightarrow (p_1)$, let $0 \le x \le y \in p_1(H \cap D_1)$. Take $(y, y') \in H \cap D_1$. Since $0 \le y' \le y' \in p_2(H \cap D_1)$, $(0, y') \in H$ by (p_2). Hence $(y, 0) = (y, y') - (0, y') \in H$. Thus $(0, 0) \le (x, 0) \le (y, 0) \in H$. Then $(x, 0) \in H$. Hence (p_1) holds. Similarly, for D_2, $(p_1) \Rightarrow (p_2)$ holds.

By a group monomorphism $f : (G, P) \to (G', P')$, we shall mean a group monomorphism f from G to G' which is *order-preserving* (that is, $f(P) \subset P'$).

For a group monomorphism $f : (G, P) \to (G', P')$, we shall say that f is a *group embedding* (or (G, P) is *group embeddable* in (G', P') via f) if f is also *order-reflecting* (that is, $f^{-1}(P') \subset P$), equivalently, $P = f^{-1}(P')$.

We note that $(\mathbb{Z}, \mathbb{Z}^*)$ is group embeddable in any (G', P') via f (defined by $f(n) = pn$ for some $p \in P'_0$).

For a group monomorphism $f : (G, P) \to (G', P')$, let $g = f \times f : (G \times G, P \times P) \to (G' \times G', P' \times P')$ be a group monomorphism defined by $g(x, y) = (f(x), f(y))$. (Evidently, $g = f \times f$ is a group embedding iff so is f).

Theorem 2.4. *For a group monomorphism $f : (G, P) \to (G', P')$, let $g = f \times f : (G \times G, P \times P) \to (G' \times G', P' \times P')$. Let T be a positive subset of $G \times G$ with $T \subset P \times P$, and let $T' = g(T)$. For a subgroup H (resp. H') of $G \times G$ (resp. $G' \times G'$), suppose (P) $g(H \cap (P \times P)) = H' \cap g(P \times P)$ holds. Then the following hold.*

(1) *H is convex for T iff H' is convex for T'.*

(2) *For each $i = 1, 2$, H satisfies (p_i) for T iff H' satisfies (p'_i) for T', here $\le\, = \,\le_{f(P)}$ in (p'_i).*

Proof. Note (T) $g(H \cap T) = H' \cap T'$ holds by (P) with $T \subset P \times P$. Thus (1) holds by Proposition 2.1, putting $h = g$. For (2), let $i = 1$. For the if part, to see (p_1), let $0 \le x \le y \in p_1(H \cap T)$. Then $0 \le f(x) \le f(y) \in p'_1(H' \cap T')$ by (T) with $f(P) \subset P'$, here $\le\, = \,\le_{f(P)}$. Since H' satisfies (p'_1), $(f(x), 0) \in H' \cap g(P \times P)$. Thus $(x, 0) \in H$ by (P). For the only if part, to see (p'_1), let $0 \le x' \le y' \in p'_1(H' \cap T')$, $\le\, = \,\le_{f(P)}$. Since $y' \in p'_1(H' \cap T')$, take $y \in p_1(H \cap T) \cap P$ with $f(y) = y'$ by (T), and $x' = f(x) \in f(P)$. But, $y' - x' = f(y - x) \in f(P)$, then $0 \le x \le y \in p_1(H \cap T)$. Thus $(x, 0) \in H$ by (p_1). Hence, $(x', 0) \in H'$ by (P). For $i = 2$, (2) is similarly shown. □

Remark 2.5. In Theorem 2.4, (P) is essential, moreover (P) can not be replaced by (T) $g(H \cap T) = H' \cap T'$ in (2) even if g is the identity map and a group embedding.

Indeed, let $f : (\mathbb{Z}, 2\mathbb{Z}^*) \to (\mathbb{Z}, 2\mathbb{Z}^*)$ be the identity map, and let $g = f \times f$, and let $H_1 = 2\mathbb{Z} \times 2\mathbb{Z}$, $H_2 = 4\mathbb{Z} \times 4\mathbb{Z}$. For (1), let $T = 2\mathbb{Z}^* \times 2\mathbb{Z}^*$. Then H_1 is convex, but H_2 is not convex for T. For (2), let $T = 4\mathbb{Z}^* \times 4\mathbb{Z}^*$. Then (T) holds, but (P) doesn't hold. Also, H_1 satisfies (p_i), but H_2 doesn't satisfy (p_i) for T. Hence, we obtain desired examples, putting $H = H_1$, $H' = H_2$ and vice versa.

3. Convexity of subgroups in the product groups

We give characterizations for subgroups of $G \times G$ to be convex for the positive subsets D_i of $G \times G$ induced by P under (G, P) being Archimedean (i.e., for each $x, y \in P_0$, $y < nx$ for some $n \in \mathbb{N}$).

Theorem 3.1. *Let (G, P) be Archimedean. For a subgroup H of $G \times G$, and the positive subsets D_i of $G \times G$ induced by P, the following hold.*

(1) *For D_0, H is convex iff $D_0 \subset H$ or $D_0 \cap H = 0$.*

(2) *For D_1, H is convex iff $D_1 \subset H$, $D_1 \cap H = P \times 0$, $D_1 \cap H = D_0$, or $D_1 \cap H = 0$.*

(3) *For D_2, H is convex iff $D_2 \subset H$, $D_2 \cap H = 0 \times P$, $D_2 \cap H = D_0$, or $D_2 \cap H = 0$.*

(4) *For D_3, H is convex iff $D_3 \subset H$, $D_3 \cap H = P \times 0$, $D_3 \cap H = 0 \times P$, or $D_3 \cap H = 0$.*

Proof. (1) The if part is obvious. For the only if part, let H be D_0-convex and $D_0 \cap H \neq 0$. Take $(p_0, p_0) \in D_0 \cap H$ with $p_0 \in P_0$. For $p \in P_0$, let $p < mp_0$ for some $m \in \mathbb{N}$. Then $(0, 0) \leq (p, p) \leq m(p_0, p_0) \in D_0 \cap H$. Thus $(p, p) \in H$. Hence $D_0 \subset H$.

(2) For the if part, let $(0, 0) \leq (x, y) \leq (x', y') \in D_1 \cap H$. For $D_1 \cap H = P \times 0$, $(x, y) = (x, 0) \in P \times 0 \subset H$, thus $(x, y) \in H$. For $D_1 \cap H = D_0$, $y' - y \leq x' - x$, but $x' = y'$, so $x \leq y$. But $y \leq x$. Then $x = y$, thus $(x, y) \in H$. Hence H is convex. For the other cases, obviously H is convex. For the only if part, let us consider the following case: (i) $D_1 \cap H = D_0 \cap H$, or (ii) $D_1 \cap H \neq D_0 \cap H$, but (ii′) $p_2(D_1 \cap H) \neq 0$ or (ii″) $p_2(D_1 \cap H) = 0$.

For (i), since H is D_1-convex and $D_0 \subset D_1$, H is D_0-convex by (i). Thus, by (1), $D_0 \subset H$ or $D_0 \cap H = 0$, which implies $D_1 \cap H = D_0$ or $D_1 \cap H = 0$ by (i).

For (ii), to see $P \times 0 \subset H$, let $p \in P_0$. Take $(t_1, t_2) \in D_1 \cap H$ with $t_1 \neq t_2$. Let $p < n(t_1 - t_2)$ for some $n \in \mathbb{N}$. Then $(0, 0) \leq (p, 0) \leq n(t_1, t_2) \in D_1 \cap H$. Since H is D_1-convex, $(p, 0) \in H$. This shows $P \times 0 \subset H$. Now, for (ii′), to see $0 \times P \subset H$, let $p \in P_0$. Take $(u_1, u_2) \in D_1 \cap H$ with $u_2 \neq 0$. Let $p < iu_2$ for some $i \in \mathbb{N}$. Then $(0, 0) \leq (p, p) \leq i(u_1, u_2) \in D_1 \cap H$. Then $(p, p) \in H$. Thus, $(0, p) = (p, p) - (p, 0) \in H$ by $P \times 0 \subset H$ in (ii). This shows $0 \times P \subset H$. Thus, $D_3 = (P \times 0) + (0 \times P) \subset H$, hence $D_1 \subset$ H. For (ii″), $D_1 \cap H \subset P \times 0$. But, $P \times 0 \subset D_1$, then $P \times 0 \subset D_1 \cap H$ by (ii). Thus $D_1 \cap H = P \times 0$.

(3) This is similarly shown as in (2), so we shall omit the proof.

(4) The if part is routinely shown. For the only if part, let us consider the following cases: (i) $p_1(D_3 \cap H) \neq 0$, $p_2(D_3 \cap H) \neq 0$ (ii) $p_1(D_3 \cap H) \neq 0$, $p_2(D_3 \cap H) = 0$ (iii) $p_1(D_3 \cap H) = 0$, $p_2(D_3 \cap H) \neq 0$ (iv) $p_1(D_3 \cap H) = 0$, $p_2(D_3 \cap H) = 0$.

For (i), to see $P \times 0 \subset H$, let $p \in P_0$. Take $(v_1, v_2) \in D_3 \cap H$ with $v_1 \neq 0$, and let $p < kv_1$ for some $k \in \mathbb{N}$. Then $(0, 0) \leq (p, 0) \leq k(v_1, v_2) \in D_3 \cap H$. Since H is D_3-convex, $(p, 0) \in H$. Hence $P \times 0 \subset H$. Similarly, $0 \times P \subset H$. Thus $D_3 \subset H$. For (ii), $P \times 0 \subset H$, and $D_3 \cap H \subset P \times 0$. Thus $D_3 \cap H = P \times 0$. For (iii), similarly $D_3 \cap H = 0 \times P$. For (iv), obviously $D_3 \cap H = 0$. □

Corollary 3.2. *Let* (R, S) *be Archimedean, in particular* $R = \mathbb{Z}$. *For an ideal* I *of the (direct) product ring* $R \times R$, *and the semi-cones* D_i *induced by* S, *the results in Theorem 3.1 remain true.*

Corollary 3.3 below is shown by (the the proof of) Theorem 3.1(2),(3) with Remark 2.3(1), here (ii) is essential in view of Remark 2.3(1). The corollary is an improvement of [9, Proposition 3.23(2)].

Corollary 3.3. *Let* (G, P) *be Archimedean. Then a subgroup of* H *of* $G \times G$ *is convex for* D_1 *(resp.* D_2*) iff (i)* H *satisfies* (p_1) *and* (p_2)*, or (ii)* $D_1 \cap H = D_0$ *(resp.* $D_2 \cap H = D_0$*).*

Remark 3.4. Let $(R, \leq)$ be a partially ordered integral domain such that $(*)$ for each non-zero element $a \in R$, $0 < a^2$ (or $0 < aa'$ for some $a' \in R$). In [3], we consider convexity of ideals of the polynomial ring $R[x]$ with the ordinary order $\leq_1$ or $\leq_2$. Note $(R[x], \leq_1)$ is *non-Archimedean*, here for $f(x) \in R[x]$, $0 <_1 f(x)$ if the leading coefficient of $f(x)$ is positive in R. Let $(R[x], S) = (R[x], \leq_1)$. Let I be an ideal of the direct product ring $R[x] \times R[x]$. Thus $I = p_1(I) \times p_2(I)$ with $p_i(I)$ ideals. For a non-zero, proper ideal I, the following hold.

(1) For D_0, I is convex iff $p_1(I) \cap p_2(I) \cap S = 0$ ($\Leftrightarrow p_1(I) = 0$ or $p_2(I) = 0$).

(2) For D_1, I is convex iff $I = R[x] \times 0$ or $I = 0 \times p_2(I)$.

(3) For D_2, I is convex iff $I = 0 \times R[x]$ or $I = p_1(I) \times 0$.

(4) For D_3, I is convex iff $I = R[x] \times 0$ or $I = 0 \times R[x]$.

Indeed, the if part is obvious. To see the only if part, let I be D_i-convex. Suppose there exists $f(x) \in p_1(I) \cap p_2(I) \cap S_0$. Then $(0, 0) \leq (1, 1) \leq (xf(x), xf(x)) \in I \cap D_i$. Thus $(1, 1) \in I$, so $I = R[x] \times R[x]$, a contradiction. Hence, $p_1(I) \cap p_2(I) \cap S_0 = \varnothing$. Next, suppose $p_1(I) \neq 0$ and $p_2(I) \neq 0$. Take $(f(x), g(x)) \in I \cap (S_0 \times S_0)$ by $(*)$ and $I = p_1(I) \times p_2(I)$. Then $f(x)g(x) \in p_1(I) \cap p_2(I) \cap S_0$, a contradiction. Thus ($p_1(I) \neq 0$, $p_2(I) = 0$), or ($p_1(I) = 0$, $p_2(I) \neq 0$). But, for an ideal $p_i(I) \neq 0$, $p_i(I)$ is S-convex iff p_i(I) $= R[x]$ (actually, assume $p_i(I)$ is S-convex. Take $f(x) \in p_i(I) \cap S_0$ by $(*)$, then $0 \leq_1 1 \leq_1 xf(x) \in p_i(I) \cap S$, thus $1 \in p_i(I)$ which yields $p_i(I) = R[x]$). Hence, (1)~(4) hold in view of the above.

4. Convexity of ideals in the product extension rings

In this section, the the symbol $R \otimes R$ means the *direct product ring* of R (as in [5, 8]), but the symbol $R \times R$ denotes the (*additive*) *group* of the ring $R \otimes R$.

The product extension ring $(R \ltimes R;\, a, b)$ is a ring which is $R \times R$ as an additive group, and the following multiplication is given (in Section 1).

$$(x_1, y_1) * (x_2, y_2) = (x_1x_2 + ay_1y_2,\, x_1y_2 + y_1x_2 + by_1y_2).$$

Note that $(R \ltimes R;\, a, b)$ has the identity $(1, 0)$, and $(0, 1) * (0, 1) = (a, b)$.

The product extension ring $(R \ltimes R;\, 0, 0)$ is denoted by $R \ltimes R$ (as in [5]).

The positive subsets D_i (except L) induced by S are semi-cones in $R \otimes R$. However, each D_i or L need not be a semi-cone in $(R \ltimes R;\, a, b)$ by the following Proposition 4.1 due to [6]. (For characterizations of semi-cones in $\mathbb{Z}$ (resp. $\mathbb{Z} \otimes \mathbb{Z}$, $\mathbb{Z} \ltimes \mathbb{Z}$), see [3] (resp. [8])).

Proposition 4.1. *For D_i induced by S of R, the following hold in $(R \ltimes R;\, a, b)$. Obviously, for $SS = 0$, every D_i except L is a semi-cone.*

(1) *D_0 is a semi-cone iff $(a + 1)SS \subset S$ and $(a - b - 1)SS = 0$.*
(2) *D_1 is a semi-cone iff $(b + 2)SS \subset S$ and $(a - b - 1)SS \subset S$.*
(3) *D_2 is a semi-cone iff $aSS \subset S$ and $(b - a)SS \subset S$.*
(4) *D_3 is a semi-cone iff $aSS \subset S$ and $bSS \subset S$.*
(5) *L is a semi-cone iff $aS = bSS = 0$, $S_0S_0 + aR \subset S_0$, and $(S_0 + bR)S \subset S$.*

Remark 4.2. (1) If L is a semi-cone in $(R \ltimes R;\, a, b)$, then $S_0S_0 \subset S_0$ (thus $SS \neq 0$), and the converse holds if $a = b = 0$. For $S \ni 1$ (resp. R being an integral domain), L is a semi-cone iff $a = b = 0$ and $S_0S_0 \subset S_0$ (resp. $a = b = 0$). For $S \ni 1$, we can not omit "$S_0S_0 \subset S_0$" (by putting $S = \mathbb{Z}^* \otimes \mathbb{Z}^*$ in $R = \mathbb{Z} \otimes \mathbb{Z}$). This suggests that we should delete "$S \ni 1$" in [6, Corollary 2.7(3)(b)].

(2) L is a cone in $(R \ltimes R;\, a, b)$ iff $a = b = 0$, S is a cone in R, and $S_0S_0 \subset S_0$ (equivalently, R is an integral domain) by (1), but L is not even a semi-cone in $R \otimes R$. Any D_i is not a cone in $(R \ltimes R;\, a, b)$ or $R \otimes R$. We note that there exist no cones in $R \otimes R$, namely, $R \otimes R$ can not be an ordered ring ([4]). A characterization for cones of $K \ltimes K$ with K a field is given in [4]. We can replace "field" by "integral domain".

In what follows, *the symbol R' means $R \ltimes R$, and the symbol P' means $0{\times}P$ in R', here P is a positive subset of R.*

Let $f' : R \to R'$ be a group monomorphism defined by $f'(x) = (0, x)$. Then $f'(P) = P'$. The symbol g' means the following group monomorphism

$$g' = f' \times f' : R \times R \to R' \times R' \text{ defined by } g'(x, y) = ((0, x), (0, y)).$$

Remark 4.3. (1) The group monomorphism $g' : (R \ltimes R;\, a, b) \to (R' \ltimes R';\, a', b')$ is never a ring homomorphism (by $g'((1, 0) * (1, 0)) \neq g'(1, 0) * g'(1, 0) = 0$).

(2) Let us define $g^* : (R \ltimes R;\, a, b) \to (R' \ltimes R';\, a', b')$ by $g^*(x, y) = ((x, 0), (y, 0))$. Then g^* is a ring monomorphism. But, for a non-zero ideal I of $(R \ltimes R;\, a, b)$, $g^*(I)$ is never an ideal of $(R' \ltimes R';\, a', b')$ (actually, for a non-zero element $(x, y) \in I$, $g^*(x, y) * ((0, 1), (0, 0)) = ((0, x), (0, y)) \notin g^*(I)$). For $g'(I)$ being an ideal, see Lemma 4.10 later.

We note that the additive group of the ring $R \otimes R$ or $(R \ltimes R;\, a, b)$ is the group $R \times R$, and so is $R' \times R'$ for $R' \otimes R'$ or $(R' \ltimes R';\, a', b')$.

Proposition 4.4. *For the group monomorphism $g' : R \times R \to R' \times R'$, let T be a positive subset of $R \times R$ (such as $T = D_i$ induced by P), and let $T' = g'(T)$. Then the following hold.*

(1) *T' is a positive subset of $R' \times R'$. Further, T' is a semi-cone of $R' \otimes R'$ as well as (any) $(R' \ltimes R'; a', b')$ with $T'T' = 0$.*

(2) *$(R \times R, T)$ is group embeddable in $(R' \times R', T')$ (in particular, let $T = P \times P$ and $T' = P' \times P'$) via $g' : (R \times R, T) \to (R' \times R', T')$.*

Proof. For (1), T' is a positive subset of $R' \times R'$ by Proposition 2.1(1), putting $h = g'$, and $T'T' = 0$ in $R' \otimes R'$ or $(R' \ltimes R'; a', b')$, noting $(0 \times R) * (0 \times R) = 0$ in R'. (2) is obvious by $T' = g'(T)$ with (1). □

Remark 4.5. (1) For the positive subsets D_i of $R \times R$ induced by P, let $D_i' = g'(D_i)$, and D_i^* be the positive subsets of $R' \times R'$ induced by P'. Then $D_i' = D_i^*$. Also, D_i' are semi-cones in (any) $(R' \ltimes R'; a', b')$ by Proposition 4. 4(1).

(2) For the positive subset L of $R \times R$ induced by P, let $L' = g'(L)$, and L^* be the positive subset of $R' \times R'$ induced by P'. Then $L' = L^* \cap g'(R \times R)$. Besides, L' is a semi-cone by Proposition 4.4(1), but L^* is not a semi-cone in (any) $(R' \ltimes R'; a', b')$ by Remark 4.2(1), noting $P' * P' = 0$.

We recall that $a \in R$ (resp. $a - b - 1 \in R$) is a unit in R iff $(0, 1)$ (resp. $(1, 1)$) is a unit in $(R \ltimes R; a, b)$.

Lemma 4.6. *Let I be an ideal of $(R \ltimes R; a, b)$. Then the following hold as the sets D_i induced by P.*

(1) $P \times 0 \subset I \Leftrightarrow D_1 \subset I \Leftrightarrow D_3 \subset I$.

(2) *If $a \in R$ is a unit,* $0 \times P \subset I \Leftrightarrow D_2 \subset I \Leftrightarrow D_3 \subset I$.

(3) *If $a - b - 1 \in R$ is a unit,* $D_0 \subset I \Leftrightarrow D_1 \subset I \Leftrightarrow D_2 \subset I \Leftrightarrow D_3 \subset I$.

Proof. For (1), assume $P \times 0 \subset I$. To see $D_3 \subset I$, let $(s, t) \in D_3$. Then $(s, 0), (t, 0) \in I$ (by $P \times 0 \subset I$). But, $(0, t) \in I$, noting $(x, 0) * (0, 1) = (0, x)$. Hence $(s, t) = (s, 0) + (0, t) \in I$. Similarly, (2) holds, noting $(x, 0) = (0, x) * (0, 1)^{-1}$, and (3) holds, noting $(x, 0) = (x, x) * (1, 1)^{-1}$. □

In Theorem 4.7 below, (1) holds by Theorem 3.1 with Lemma 4.6. (2) holds by Proposition 4.4(1) with (1), noting (R', P') is Archimedean. (1) is a generalization of [9, Theorem 4.5], where I is generated by a single element in $(\mathbb{Z} \ltimes \mathbb{Z}; a, b)$.

Theorem 4.7. *The following hold.*

(1) *Let (R, S) be Archimedean. For an ideal I of $(R \ltimes R; a, b)$, the following hold, but we assume D_i are semi-cones induced by S.*

(a) *For D_0, I is convex iff $D_0 \subset I$ or $D_0 \cap I = 0$.*

(b) *For D_1, I is convex iff $D_1 \subset I$, $D_1 \cap I = D_0$, or $D_1 \cap I = 0$.*

(c) *For D_2, I is convex iff $D_2 \subset I$, $D_2 \cap I = 0 \times S$, $D_2 \cap I = D_0$, or $D_2 \cap I = 0$.*

(d) *For D_3, I is convex iff $D_3 \subset I$, $D_3 \cap I = 0 \times S$, or $D_3 \cap I = 0$.*

(For $a \in R$ being a unit, we can delete $D_2 \cap I = 0{\times}S$ in (c), and $D_3 \cap I = 0{\times}S$ in (d). For $a{-}b{-}1 \in R$ being a unit, we can delete $D_1 \cap I = D_0$ in (b), and $D_2 \cap I = D_0$ in (c)).

(2) *Let (R, P) be Archimedean. Let D_i be the positive subsets of $R \times R$ induced by P. Then, for an ideal I' and semi-cones $D_i' = g'(D_i)$ induced by P' in $(R' \ltimes R'; a', b')$, the results in (1) remain true, replacing "S" by "P", and adding the prime " ' " on the symbols (such as $a' \in R'$).*

Remark 4.8. In Theorem 4.7, let $I_1 = I \cap (S \times 0)$, $I_2 = I \cap (0 \times S)$. Then we have the following in $(R \ltimes R; a, b)$ and its (systematic) analogue in $(R' \ltimes R'; a', b')$ (under the same assumptions there).

(1) For D_1, I is convex with $I_1 \neq 0$ iff $D_1 \subset I$.

(2) For D_2, I is convex with $I_2 \neq 0$ iff $D_2 \subset I$ or $D_2 \cap I = 0 \times S$. (For $a \in R$ being a unit, $D_2 \cap I = 0 \times S$ is deleted).

(3) For D_3, I is convex with $I_1 \neq 0$ iff $D_3 \subset I$.

(For $R = \mathbb{Z}$, $I_1 \neq 0$ (resp. $I_2 \neq 0$) iff $I_1' = I \cap (\mathbb{Z} \times 0) \neq 0$ (resp. $I_2' = I \cap (0 \times \mathbb{Z}) \neq 0$) (indeed, for $I_1' \neq 0$, take $m, n \in \mathbb{N}$ with $(m, 0) \in I$ and $n \in S_0$, then $(mn, 0) \in I_1$, thus $I_1 \neq 0$)).

Remark 4.9. The following in [9, Proposition 3.11] is also shown by Remark 4.8(2), and its (systematic) analogue in $(R' \ltimes R'; a', b')$ holds under $a' = 1'$.

(Proposition) *For an ideal I of $(\mathbb{Z} \ltimes \mathbb{Z}; a, b)$ with $a = 1 \leq b$ (hence D_2 is a semi-cone by Proposition 4.1), I is D_2-convex with $I_2' = I \cap (0 \times \mathbb{Z}) \neq 0$ iff $D_2 \subset I$.*

$a = 1$ is essential for the only if part (by an ideal $I = 0 \times \mathbb{Z}$ of $(\mathbb{Z} \ltimes \mathbb{Z}; a, b)$ with $a = 0 \leq b$), and $I_2' \neq 0$ is also essential by $I = 0$ (cf. [9, Remark 3.12(1)]). We note that there exist no examples of $I \neq 0$ under $(*)$ $a = 1$, $b \neq 0$ by the following fact.

(Fact) *For an ideal I of $(\mathbb{Z} \ltimes \mathbb{Z}; a, b)$ with $(*)$, $I_2' = 0$ iff $I = 0$ ($\Leftrightarrow p_2(I_2') = 0$).*

(Indeed, the if part is clear, so assume $I_2' = 0$. Since $\mathbb{Z}$ is a principal ideal domain, $I = (m, k) * (\mathbb{Z} \times 0) + (0, n) * (\mathbb{Z} \times 0)$ for some $m, n, k \in \mathbb{Z}$ by [5, Proposition 3.8]. Then $n = 0$ by $I_2' = 0$. Since $(m, k) * (0, 1) \in I$ and $a = 1$, we have $k = mx$, $m + bk = kx$ for some $x \in \mathbb{Z}$. Then $m(x^2 - bx - 1) = 0$. But $d = x^2 - bx - 1 \neq 0$ by $b \neq 0$, noting for $d = 0$, $2x = (b \pm \sqrt{b^2 + 4}\,) \notin \mathbb{Z}$. Thus $m = 0$, and $k = 0$. Hence $I = 0$).

Lemma 4.10. *Let $g' : (R \ltimes R; a, b) \to (R' \ltimes R'; a', b')$ be the group monomorphism with $a' = (c, d)$, $b' = (e, f) \in R'$. Let I be an ideal of $(R \ltimes R; a, b)$, and $I^* = g'(I)$* (thus $I^* * I^* = 0$)*. Then I^* is a (proper) ideal of $(R' \ltimes R'; a', b')$ iff (C) $((a - c)y, (b - e)y) \in I$ for any $y \in p_2(I)$ ($\Leftrightarrow (cy, x + ey) \in I$ for any $(x, y) \in I$) holds. Thus, I^* is an ideal for $(a - c, b - e) \in I$ (specially, $c = a$ and $e = b$).*

Proof. Obviously, $I^* * I^* = 0$. For $x, y, x_i, y_i \in R$,

$$\begin{aligned} g'(x, y) * ((x_1, x_2), (y_1, y_2)) &= ((0, x), (0, y)) * ((x_1, x_2), (y_1, y_2)) \\ &= ((0, z_1), (0, z_2)) = g'(z_1, z_2) \end{aligned}$$

in $(R' \ltimes R'; a', b')$, here $z_1 = xx_1 + cyy_1$, $z_2 = yx_1 + xy_1 + eyy_1$. Also, for $(x, y) \in I$, $(x, y) * (x_1, y_1) = (z_1, z_2) + ((a - c)yy_1, (b - e)yy_1) \in I$. Since I is an ideal, $g'(I)$ is an ideal $\Leftrightarrow$ $g'(z_1, z_2) \in g'(I)$ (i.e., $(z_1, z_2) \in I$) for any $(x, y) \in I$, $x_1, y_1 \in R$ $\Leftrightarrow$ $((a - c)yy_1, (b - e)yy_1) \in I$ for any $y_1 \in R$, $y \in p_2(I)$ $\Leftrightarrow$ (C). For the parenthetic part, note $(cy, x + ey) = (x, y) * (0, 1) - ((a - c)y, (b - e)y)$ in $(R \ltimes R; a, b)$. □

For a finitely generated ideal $I_0 = \sum_{i=1}^{n} (u_i, v_i) * (R \ltimes R; a, b)$ of $(R \ltimes R; a, b)$ with $u_i, v_i \in R$, let us define the following finitely generated ideals of $(R' \ltimes R'; a', b')$:

$I_0^* = \sum_{i=1}^{n} (u_i', v_i') * (R' \ltimes R'; a', b')$, where $u_i' = (0, u_i)$, $v_i' = (0, v_i)$, $a' = (a, c)$, $b' = (b, d) \in R'$ $(c, d \in R)$.
$I_0' = \sum_{i=1}^{n} (u_i', v_i') * (R' \ltimes R'; a', b')$, where $u_i' = (u_i, 0)$, $v_i' = (v_i, 0)$, $a' = (a, 0)$, $b' = (b, 0) \in R'$.

Hereafter, *the symbols I_0, I_0^*, and I_0' mean these finitely generated ideals.*

Lemma 4.11. *Let $g' : (R \ltimes R; a, b) \to (R' \ltimes R'; a', b')$ be the group monomorphism. Then the following hold for I_0 in $(R \ltimes R; a, b)$*, and I_0^*, I_0' in $(R' \ltimes R'; a', b')$.

(1) $I_0^* = g'(I_0)$ *under $a' = (a, c)$, $b' = (b, d) \in R'$ $(c, d \in R)$.*

(2) $I_0' \cap (R_0 \times R_0) = g'(I_0)$ *under $a' = (a, 0)$, $b' = (b, 0) \in R'$, here $R_0 = 0 \times R$.*

Proof. (1) holds, noting the following holds in $(R'\ltimes R'; a', b')$.

$$((0, u), (0, v)) * ((x_1, x_2), (y_1, y_2)) = ((0, z_1), (0, z_2)) = g'((u, v) * (x_1, y_1)),$$

here $z_1 = ux_1 + avy_1$, $z_2 = vx_1 + (u + bv)y_1$ as in the proof of Lemma 4.10.

For (2), the following $(*)$ and $(**)$ hold in $(R'\ltimes R'; a', b')$.

$$(*)\qquad ((u, 0), (v, 0)) * ((x_1, x_2), (y_1, y_2)) = ((z_1, z_2), (w_1, w_2)),$$

$$(**)\qquad ((u, 0), (v,0)) * ((0, x_2), (0, y_2)) = ((0, z_2), (0, w_2)) = g'((u, v) * (x_2, y_2)),$$

here $z_1 = ux_1 + avy_1$, $z_2 = ux_2 + avy_2$, $w_1 = vx_1 + (u + bv)y_1$, $w_2 = vx_2 + (u + bv)y_2$.
Then (2) holds (indeed, $g'(I_0) \subset I_0' \cap (R_0 \times R_0)$ by $(**)$. For $I_0' \cap (R_0 \times R_0) \subset g'(I_0)$, let $((0, x), (0, y)) = \sum_{i=1}^{n} ((u_i, 0), (v_i, 0)) * ((x_{i1}, x_{i2}), (y_{i1}, y_{i2})) \in I_0'$. Then $x = \sum_{i=1}^{n} z_{i2}$, $y = \sum_{i=1}^{n} w_{i2}$ by $(*)$, here $z_{i2} = u_i x_{i2} + av_i y_{i2}$, $w_{i2} = v_i x_{i2} + (u_i + bv_i)y_{i2}$. Thus $((0, x), (0, y)) = \sum_{i=1}^{n} g'((u_i, v_i) * (x_{i2}, y_{i2})) \in g'(I_0)$ by $(**)$). □

For convexity of an ideal I of $(R \ltimes R; a, b)$ for D_i induced by S, we assume D_i are semi-cones in [7, 9] (in view of Proposition 4.1), but we assume the following.

We consider an ideal I (resp. subsets D_i induced by S) in $(R \ltimes R; a, b)$ as a subgroup (resp. the positive subsets induced by P) in the (additive) group $R \times R$ of $(R \ltimes R; a, b)$, unless otherwise stated.

Theorem 4.12. *For the group embedding* $g' : ((R \ltimes R; a, b), P \times P) \to ((R' \ltimes R'; a', b'), P' \times P')$, *let* $D_i' = g'(D_i)$. *Then the following hold.*

(1) D_i' *are semi-cones in* $(R'\ltimes R'; a', b')$ *induced by* P' *with* $D_i' * D_i' = 0$. *Moreover,* $((R \ltimes R; a, b), D_i)$ *are group embeddable in* $((R'\ltimes R'; a', b'), D_i')$ *via* $g' : ((R \ltimes R; a, b), D_i) \to ((R'\ltimes R'; a', b'), D_i')$.

(2) *For an ideal* I *of* $(R \ltimes R; a, b)$, *suppose* $I^* = g'(I)$ *is an ideal of* $(R'\ltimes R'; a', b')$ *with* $a' = (c, d)$, $b' = (e, f) \in R'$ (*equivalently,* $((a - c)y, (b - e)y) \in I$ *for any* $y \in p_2(I)$). *Then the following* (*a*) *and* (*b*) *hold.*

(a) I *is convex for* D_i *iff* I^* *is convex for* D_i'.

(b) I *satisfies* (p_1) (*resp.* (p_2)) *for* D_i *iff* I^* *satisfies* (p_1') (*resp.* (p_2')) *for* D_i', *here* $\leq = \leq_{P'}$ *in* (p_i').

(3) *Let* I_0 *in* $(R \ltimes R; a, b)$. *For* I_0^* (*resp.* I_0') *in* $(R' \ltimes R'; a', b')$, *let* $a' = (a, c)$, $b' = (b, d) \in R'$ $(c, d \in R)$ (*resp.* $a' = (a, 0)$, $b' = (b, 0) \in R'$). *Then for* I_0, *and* I_0^* (*resp.* I_0'), (*a*) *and* (*b*) *in* (2) *also hold.*

Proof. (1) holds by Proposition 4.4. (2) holds by Theorem 2.4 with Lemma 4.10, noting $(*)$ $g'(I \cap (P \times P)) = I^* \cap g'(P \times P)$, and $f'(P) = P'$ in (b). For (3), in $(*)$ we can put $I = I_0$, and $I^* = I_0^*$ or I_0' by Lemma 4.11. □

Related to Theorem 4.12(2),(3), let us give the following example.

Example 4.13. (1) For I_0 in $(\mathbb{Z} \ltimes \mathbb{Z}; a, b)$, $g'(I_0)$ need not be an ideal of $(\mathbb{Z}' \ltimes \mathbb{Z}'; a', b')$ (indeed, let $I_0 = (2, 1) * (\mathbb{Z} \ltimes \mathbb{Z}) = \{(2x, x + 2y) \mid x, y \in \mathbb{Z}\}$, and $a' = b' = 1' = (1, 0) \in R'$. Then $((0, 2), (0, 1)) \in g'(I_0)$, but $((0, 2), (0, 1)) * (0, 1') = ((0, 1), (0, 3)) \notin g'(I_0)$).

(2) Let us give examples to (i) ~ (iv) for I_0, and the positive subsets D_i in the group $\mathbb{Z} \times \mathbb{Z}$ of $(\mathbb{Z} \ltimes \mathbb{Z}; a, b)$ below. Then, by Theorem 4.12(3), we have the similar examples to (i) ~ (iv) for I_0^* (resp. I_0') and semi-cones $D_i' = g'(D_i)$ in $(\mathbb{Z}' \ltimes \mathbb{Z}'; a', b')$, here $a' = (a, c)$, $b' = (b, d) \in \mathbb{Z}'$ $(c, d \in \mathbb{Z})$ (resp. $a' = (a, 0)$, $b' = (b, 0) \in \mathbb{Z}'$)

(i) I_0 is D_0-convex, but I_0 is neither D_1-convex nor D_2-convex. (ii) I_0 is D_1-convex, but I_0 is not D_2-convex. (iii) I_0 is D_2-convex, but I_0 is not D_1-convex. (iv) I_0 is D_3-convex, thus D_i-convex. (Obviously, I_0 is D_0-convex in (ii), (iii), but I_0 is not

D_3-convex in (i), (ii), (iii)).

To see (i) ~ (iv), let D_i be the positive subsets induced by $P = 2\mathbb{Z}^*$.

For (i), let $I_0 = (1, 1) * (\mathbb{Z} \ltimes \mathbb{Z}; -2, 2) = \{(x - 2y, x + 3y) \mid x, y \in \mathbb{Z}\}$. Then I_0 is D_0-convex by $D_0 \subset I_0$. But, I_0 is not D_1-convex (by $(0, 0) \leq (2, 0) \leq (10, 0) \in I_0$, but $(2, 0) \notin I_0$). Similarly, I_0 is not D_2-convex.

For (ii), let $I_0 = (0, 4) * (\mathbb{Z} \ltimes \mathbb{Z}; 0, -3) = \{(0, 4x) \mid x \in \mathbb{Z}\}$. Then I_0 is D_1-convex by $I_0 \cap D_1 = 0$, but I_0 is not D_2-convex (by $(0, 0) \leq (0, 2) \leq (0, 4) \in I_0$, but $(0, 2) \notin I_0$).

For (iii), let $I_0 = (5, 1) * (\mathbb{Z} \ltimes \mathbb{Z}; 5, -4) = \{(5x, x) \mid x \in \mathbb{Z}\}$. Then I_0 is D_2-convex by $I_0 \cap D_2 = 0$, but I_0 is not D_1-convex (by $(0, 0) \leq (2, 0) \leq (10, 2) \in I_0$, but $(2, 0) \notin I_0$).

For (iv), let $I_0 = (1, 1) * (\mathbb{Z} \ltimes \mathbb{Z}; 0, -3) = \{(x, x - 2y) \mid x, y \in \mathbb{Z}\}$. Then I_0 is D_3-convex by $I_0 \supset D_3$.

In the above (i) ~ (iv), any D_i induced by $S = 2\mathbb{Z}^*$ is not a semi-cone in the respective ring $(\mathbb{Z} \ltimes \mathbb{Z}; a, b)$ (actually, for $(2, 2) \in D_i$, $(2, 2) * (2, 2) \notin D_i$).

Finally, in terms of condition (p_1) or (p_2), we give the following observation. (1) is shown in [7, 9], but (b)(ii) holds by Corollary 3.3 with (b)(i). (2), (3) hold by Theorem 4.12.

Observation 4.14. (1) The following hold in $(R \ltimes R; a, b)$, but assume that D_i are semi-cones induced by S.

(a) Let I be an ideal of $(R \ltimes R; a, b)$ with $I = p_1(I) \times p_2(I)$. For D_i, I satisfies (p_1) and (p_2) iff I is convex. (We note that for an ideal I of $R \otimes R$, $I = p_1(I) \times p_2(I)$, D_i induced by S are semi-cones, and the above result remains true in $R \otimes R$).

(b) Let I be an ideal of $(R \ltimes R; a, b)$. Then the following hold.

(i) For D_0 (resp. D_1; D_2), if I satisfies (p_1) (resp. (p_1); (p_1) and (p_2)), then I is convex. These converses hold if $a - b - 1 \in R$ is a unit, but assume $SS = 0$ for D_0 (for other conditions to the converses for D_1 or D_2, see [9, Proposition 3.21]). For D_3, I satisfies (p_1) and (p_2) iff I is convex. We note (i′), (ii′), and (iii′) below.

(i′) For D_0 or D_1, (p_1) implies (p_2). (ii′) For D_2 or D_3, (p_1) need not imply (p_2) or convexity of I. (iii′) For each D_i, (p_2) need not imply (p_1) or convexity of I.

(Indeed, for (i′), to see (p_2), let $0 \leq x \leq y \in p_2(I \cap D_i)$ $(i = 0, 1)$. Then $0 \leq x \leq x' \in p_1(I \cap D_i)$ for some $(x', y) \in I \cap D_i$. Thus $(x, 0) \in I$ by (p_1), so $(0, x) = (x, 0) * (0, 1) \in I$. For (ii′) (resp. (iii′)), consider an ideal $I = 0 \times 4\mathbb{Z}$ (resp. $I = 4\mathbb{Z} \times \mathbb{Z}$) and D_i induced by $S = 2\mathbb{Z}^*$ in $(\mathbb{Z} \ltimes \mathbb{Z}; 0, b)$ (here, D_0, D_1 are semi-cones for $b = -1$, and so are D_2, D_3 for $b \in \mathbb{Z}^*$ by Proposition 4.1)).

(ii) Suppose (R, S) is Archimedean. Then, for D_1 (resp. D_2), I is convex iff I satisfies (p_1) or $D_1 \cap I = D_0$ (resp. (p_1) and (p_2), or $D_2 \cap I = D_0$).

(2) For an ideal I' and semi-cones $D_i' = g'(D_i)$ in $(R' \ltimes R'; a', b')$, the results in (1) remain true, replacing " S " by " P ", and adding the prime " ′ " on the symbols, here we can delete "$SS = 0$ for D_0" in (b)(i). (For an ideal I' of $R' \otimes R'$, the similar holds for the parenthetic part in (a) by Proposition 4.4(1)).

(3) Let I be an ideal of $(R \ltimes R; a, b)$. For I_0^* (resp. I_0') in $(R' \ltimes R'; a', b')$, let $a' = (a, c)$, $b' = (b, d) \in R'$ $(c, d \in R)$ (resp. $a' = (a, 0)$, $b' = (b, 0) \in R'$). Suppose $I^* = g'(I)$ is an ideal of $(R' \ltimes R'; a', b')$, in particular $I^* = I^*{}_0 = g'(I_0)$. Then I (resp. I_0) satisfies (p_j) for D_i iff I^* (resp. I_0') satisfies (p_j') for D_i' $(= g'(D_i))$, here $j = 1, 2$. Also, applying this to (2), we have the following:

For D_1, if I (resp. I_0) satisfies (p_1), then I^* (resp. I_0') is convex for D_1'. The converse holds if $a' - b' - 1' \in R'$ is a unit, or $D_1' \cap I^* \neq D_0'$ (resp. $D_1' \cap I_0' \neq D_0'$) with (R', P') Archimedean. For the other D_i, some applications to (2) will be similarly obtained.

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積拡大環における凸イデアル

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要　　旨

半順序環における凸イデアルは，それらの剰余環に自然に誘導された半順序を与える ([1])。半順序群における凸部分群に対しても同様である。環や群における半順序はそれぞれ半コーン ([2,3]) や正集合 ([4]) によって決定される。本稿では，標準的な正集合をもつ直積群における部分群の凸性について，特徴付けを与える。さらに，それらの正集合に類似型の半コーンをもつ積拡大環の構成方法を与え，そこにおけるイデアルの凸性を考察する。

キーワード: 直積群，積拡大環，半順序，正集合，半コーン，凸部分群，凸イデアル，単射群準同型

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