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Exponential Laws and Ring Morphisms

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Abstract

Let $\mathbb{R}_+$ be the set of positive real numbers. We show that there is a one to one correspondence between the set of 2 variable functions $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+$ satisfying the exponential laws and the set of ring morphisms from $\mathbb{R}$ to the ring of $\mathbb{Q}$-linear transformations of $\mathbb{R}$. As an application, we show that there is a non-constant 2 variable function $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+$ satisfying the exponential laws such that $f(x, y)$ is not equal to x^y as functions.

Keywords: exponential laws, ring morphisms

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1. Introduction

It is well known that the exponential function $x^y:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+$ satisfies the following exponential laws

$$(1)\ (x_1x_2)^y = x_1{}^y x_2{}^y \quad (2)\ x^{y1+y2} = x^{y1}x^{y2} \quad (3)\ (x^y)^z = x^{yz}$$

where $\mathbb{R}_+$ is the set of positive real numbers. Is the exponential function $x^y:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+$ a unique function satisfying the exponential laws? That is to say, let $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+$ be a non-constant 2-variable function satisfying

(Ex1) $f(x_1x_2, y) = f(x_1, y)f(x_2, y)$
(Ex2) $f(x, y_1 + y_2) = f(x, y_1)f(x, y_2)$
(Ex3) $f(f(x, y), z) = f(x, yz)$

Whether $f(x, y) = x^y$ or not? In the case of 1-variable functions $g:\mathbb{R}\to\mathbb{R}_+$ satisfying $g(x + y) = g(x)g(y)$ for any $x, y\in\mathbb{R}$, it is well known that there is such a function $g(x)$ which is not equal to a^x for any $a\in\mathbb{R}$ (see e.g. [3]). In this paper, we show that 2-variable functions satisfying the conditions (Ex1), (Ex2) and (Ex3) correspond to ring morphisms from $\mathbb{R}$ to the

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endomorphism ring $\mathrm{End}_{\mathbb{Q}}(\mathbb{R})$ of $\mathbb{Q}$-linear transformations of $\mathbb{R}$. As an application, we show that $f(x, y)$ is not equal to x^y in general.

2. Exponential Laws and Bilinear Functions

We start to consider bilinear functions induced by functions satisfying exponential laws.

Proposition 1. *Let $\mathcal{E}$ be the set of 2-variable functions $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ satisfying the conditions* (Ex1), (Ex2) *and* (Ex3), *and let $\mathcal{B}$ be the set of 2-variable functions $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:*

(B1) $g(x_1 + x_2, y) = g(x_1, y) + g(x_2, y)$
(B2) $g(x, y_1 + y_2) = g(x, y_1) + g(x, y_2)$
(B3) $g(g(x, y), z) = g(x, yz)$

For $f \in \mathcal{E}$, we set $\tilde{f}(x,y) = \log f(e^x, y)$ for any $x, y \in \mathbb{R}$, and for $g \in \mathcal{B}$, we set $\bar{g}(x,y) = e^{g(\log x, y)}$ for any $x \in \mathbb{R}_+$ and $y \in \mathbb{R}$. Then the operations $\sim$ and $-$ induce a one to one correspondence between $\mathcal{E}$ and $\mathcal{B}$. Especially, for $f \in \mathcal{E}$, $f(x, y) = x^y$ if and only if $\tilde{f}(x,y) = xy$.

Proof. For $f \in \mathcal{E}$, we have

$$\begin{aligned}\tilde{f}(x_1+x_2,y) &= \log f(e^{x_1+x_2},y)\\ &= \log f(e^{x_1}e^{x_2},y)\\ &= \log\left(f(e^{x_1},y)f(e^{x_2},y)\right)\\ &= \log f(e^{x_1},y)+\log f(e^{x_2},y)\\ &= \tilde{f}(x_1,y)+\tilde{f}(x_2,y)\end{aligned}$$

$$\begin{aligned}\tilde{f}(x,y_1+y_2) &= \log f(x,y_1+y_2)\\ &= \log\left(f(e^x,y_1)f(e^x,y_2)\right)\\ &= \log f(e^x,y_1)+\log f(e^x,y_2)\\ &= \tilde{f}(x,y_1)+\tilde{f}(x,y_2)\end{aligned}$$

$$\begin{aligned}\tilde{f}(\tilde{f}(x,y),z) &= \tilde{f}(\log f(e^x,y),z)\\ &= \log f(e^{\log f(e^x,y)},z))\\ &= \log f(f(e^x,y),z)\\ &= \log f(e^x,yz)\\ &= \tilde{f}(x,yz)\end{aligned}$$

Therefore, $\tilde{f}$ satisfies the conditions (B1), (B2), and (B3), and it belongs to $\mathcal{B}$. Similarly, for $g \in \mathcal{B}$, $\bar{g}$ belongs to $\mathcal{E}$. Furthermore, we have

$$\begin{aligned}\bar{\tilde{f}}(x,y) &= e^{\tilde{f}(\log x,y)}\\ &= e^{\log f(e^{\log x},y)}\\ &= f(x,y)\end{aligned} \qquad \begin{aligned}\tilde{\bar{g}}(x,y) &= \log \bar{g}(e^x,y)\\ &= \log e^{g(\log e^x,y)}\\ &= g(x,y)\end{aligned}$$

Hence the operations $\sim$ and $-$ induce a one to one pondance between $\mathcal{E}$ and $\mathcal{B}$. In the case $f(x, y) = x^y$, we have

$$\begin{aligned}\tilde{f}(x,y) &= \log f(e^x, y)\\ &= \log (e^x)^y\\ &= \log e^{xy}\\ &= xy\end{aligned}$$

□

Corollary 2. *Let* $f:\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ *be a non-constant 2-variable function satisfying the conditions* (Ex1), (Ex2) *and* (Ex3). *If* $f(x, y)$ *is continuous, then* $f(x, y) = x^y$ *as functions.*

Proof. Since $\log x$ and e^x is continuous, by Proposition 1 it suffices to show that if $g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfying the conditions (B1), (B2) and (B3) is continuous and is not constant, then $g(x, y) = xy$. Let $g(1, 1) = a$, then the conditions (B1) and (B2) say that $g(m, n) = amn$ for any integers m, n. Then it is easy to see that $g(p, q) = apq$ for any $p, q \in \mathbb{Q}$. Since g is continuous, $g(x, y) = axy$ for any $x, y \in \mathbb{R}$. By the condition (B3), we have

$$\begin{aligned} g(g(1,1),1) &= g(1,1)\\ a^2 &= a\\ a &= 0,1\end{aligned}$$

In the case $a = 0$, g is constant. Henece we have $a = 1$, $g(x, y) = xy$. □

3. Exponential Laws and Ring Morphisms

We denote by Fun($\mathbb{R}$, $\mathbb{R}$) the set of functions from $\mathbb{R}$ to $\mathbb{R}$. A mapping $h : \mathbb{R} \to \mathrm{Fun}(\mathbb{R}, \mathbb{R})$ means that for any $y \in \mathbb{R}$, $h(y):\mathbb{R}\to\mathbb{R}$ is a function with $h(y)(x) \in \mathbb{R}$ for any $x \in \mathbb{R}$.

Proposition 3. *Let* $\mathcal{T}$ *be the set of mappings* $h : \mathbb{R} \to \mathrm{Fun}(\mathbb{R}, \mathbb{R})$ *satisfying the following conditions:*

(L1) $h(y)(x_1 + x_2) = h(y)(x_1) + h(y)(x_2)$
(L2) $h(y_1 + y_2)(x) = h(y_1)(x) + h(y_2)(x)$
(L3) $(h(y) \circ h(z))(x) = h(yz)(x)$

For $g \in \mathcal{B}$, *we set* $\hat{g}(y)(x) = g(x,y)$ *for any* $x, y \in \mathbb{R}$, *and for* $h \in \mathcal{T}$, *we set* $\check{h}(x,y) = h(y)(x)$ *for any* $x, y \in \mathbb{R}$. *Then the operations* $\wedge$ *and* $\vee$ *induce a one to one correspondence between* $\mathcal{B}$ *and* $\mathcal{T}$. *Especially, for* $g \in \mathcal{B}$, $g(x,y) = xy$ *if and only if* $\hat{g}(y)(x) = xy$.

Proof. For $g \in \mathcal{B}$, it is easy to see that $\hat{g}$ satisfies the conditions (L1) and (L2). According to the condition (B3), for any $x, y \in \mathbb{R}$, we have

$$\begin{aligned}(\hat{g}(y)\circ\hat{g}(z))(x) &= \hat{g}(y)(\hat{g}(z)(x))\\ &= \hat{g}(y)(g(x,z))\\ &= g(g(x,z),y)\\ &= g(x,yz)\\ &= \hat{g}(yz)(x)\end{aligned}$$

Therefore $\hat{g}$ satisfies the condition (L3). Similarly, for $h \in \mathcal{T}$, $\check{h}$ belongs to $\mathcal{B}$. Furthermore, we have

$$\begin{aligned}\check{\hat{g}}(x,y) &= \hat{g}(y)(x) \qquad & \hat{\check{h}}(y)(x) &= \check{h}(x,y)\\ &= g(x,y) & &= h(y)(x)\end{aligned}$$

Hence the operations $\wedge$ and $\vee$ induce a one to one correspondence between $\mathcal{B}$ and $\mathcal{T}$. In the case $g(x, y) = xy$, it is clear that $\hat{g}(y)(x) = xy$. □

Definition 4. *A non-empty set R is called a ring provided that there are "zero element" 0 and an addition operation* $+ : R\times R \to R \quad ((a,b)\mapsto a+b)$ *and a multiplication operation* $\cdot : R\times R\to R \quad ((a,b)\mapsto ab)$ *such that*

(1) $a+(b+c)=(a+b)+c$ *for any* $a,b,c\in R$,

(2) $a+0=0+a=a$ *for any* $a\in R$,

(3) *for any* $a\in R$ *there is* $b\in R$ *such that* $a+b=0$,

(4) $a+b=b+a$ *for any* $a,b,c\in R$,

(5) $a(bc)=(ab)c$ *for any* $a,b,c\in R$,

(6) $a(b+c)=ab+ac,\ (b+c)a=ba+ca$ *for any* $a,b,c\in R$.

For two rings R and R′, a mapping $\varphi : R\to R'$ *is called a ring morphism if* φ satisfies

(1) $\varphi(a+b)=\varphi(a)+\varphi(b)$ *for any* $a,b\in R$,

(2) $\varphi(ab)=\varphi(a)\varphi(b)$ *for any* $a,b\in R$.

Remark 5. *Usually, R is assumed to have identity* 1, *that is* $a\cdot 1=1\cdot a=a$ *for any* $a\in R$, *and a ring morphism* $\varphi : R\to R'$ *is assumed to satisfy* $\varphi(1)=1$ *(see e.g.* [1]*). But, in this paper we don't assume that identity exists and that a ring mophism preserves identity.*

Example 6. *Let V be a* $\mathbb{Q}$*-vector space,* $\mathrm{End}_{\mathbb{Q}}(V)$ *the set of* $\mathbb{Q}$*-linear transformations of V. For* $\alpha,\beta\in\mathrm{End}_{\mathbb{Q}}(V)$, *we define an addition operation* $\alpha+\beta$ *and a multiplication operation* $\alpha\beta$ *by*

$$(\alpha+\beta)(v)=\alpha(v)+\beta(v)$$
$$(\alpha\beta)(v)=\alpha(\beta(v))$$

for any $v\in V$. *Then* $\alpha+\beta$ *and* $\alpha\beta$ *are also* $\mathbb{Q}$*-linear transformations, and hence* $\mathrm{End}_{\mathbb{Q}}(V)$ *is a ring.*

The following lemma is well known as the fact that the set of 1-variable functions $f:\mathbb{R}\to\mathbb{R}$ satisfying $f(x+y)=f(x)+f(y)$ for any $x,y\in\mathbb{R}$ coincides with the endomorphism ring $\mathrm{End}_{\mathbb{Q}}(\mathbb{R})$.

Lemma 7. *If a function* $\alpha:\mathbb{R}\to\mathbb{R}$ *satisfies*

$$\alpha(x+y)=\alpha(x)+\alpha(y)$$

for any $x,y\in\mathbb{R}$, *then* α *is a* $\mathbb{Q}$*-linear transformation.*

proof. For any integer n and $x\in\mathbb{R}$, we have $\alpha(nx)=n\alpha(x)$. Then it is easy to see that $\alpha(qx)=q\alpha(x)$ for any $q\in\mathbb{Q}$ and $x\in\mathbb{R}$. □

Lemma 8. *The set* $\mathcal{T}$ *in Proposition 3 coincides with the set of ring morphisms from* $\mathbb{R}$ *to* $\mathrm{End}_{\mathbb{Q}}(\mathbb{R})$.

proof. Let $h:\mathbb{R}\to\mathrm{Fun}(\mathbb{R},\mathbb{R})$ be a mapping satisfying the conditions (L1), (L2) and (L3). By Lemma 7, $h(y)$ is a $\mathbb{Q}$-linear transformation of $\mathbb{R}$ for any $y\in\mathbb{R}$. This means $h:\mathbb{R}\to\mathrm{End}_{\mathbb{Q}}(\mathbb{R})$. According to the conditions (L2) and (L3), h is a ring morphism from $\mathbb{R}$ to $\mathrm{End}_{\mathbb{Q}}(\mathbb{R})$.

□

Theorem 9. *For $f \in \mathcal{E}$, by taking $\hat{\hat{f}}(y)(x) = \log f(e^x, y)$, $\hat{\hat{f}}$ is a ring morphism from $\mathbb{R}$ to $\mathrm{End}_{\mathbb{Q}}(\mathbb{R})$. For a ring morphism $h : \mathbb{R} \to \mathrm{End}_{\mathbb{Q}}(\mathbb{R})$, by taking $\bar{\bar{h}}(x,y) = e^{h(y)(\log x)}$, $\bar{\bar{h}}$ belongs to $\mathcal{E}$. The operations* $\hat{\smile}$ and $\bar{\vee}$ *induce a one to one correspondence between $\mathcal{E}$ and the set $\mathcal{T}$ of ring morphisms from $\mathbb{R}$ to $\mathrm{End}_{\mathbb{Q}}(\mathbb{R})$. Especially, for $f \in \mathcal{E}, f(x, y) = x^y$ if and only if $\hat{\hat{f}}(y)(x) = yx$.*

proof. By Propositions 1, 3, and Lemma 8, it is trivial. □

Corollary 10. *There exists a non-constant 2-variable function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ satisfying the conditions* (Ex1), (Ex2) *and* (Ex2) *such that $f(x, y)$ is not equal to x^y as functions.*

proof. According to the axiom of choice, there exists a $\mathbb{Q}$-vector basis $\{v_i\}_{i\in\Lambda}$ of $\mathbb{R}$ such that the cardinality of Λ is an infinite cardinal (see e.g. [3], [4]). By basic results of set theory, there exist subsets Γ and $\{v\}$ of Λ such that the cardinality of Γ is equal to one of Λ and that Λ is a disjoint union of Γ and $\{v\}$ (see e.g. [2]). Therefore, we have $\mathbb{Q}$-linear maps

$$\begin{array}{cc} (\alpha_1, \alpha_2) : \mathbb{R} \oplus \mathbb{Q} \to \mathbb{R} \quad , & \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} : \mathbb{R} \to \mathbb{R} \oplus \mathbb{Q} \\ (x_1 \oplus x_2 \mapsto \alpha_1(x_1) + \alpha_2(x_2)) & (x \mapsto \beta_1(x) \oplus \beta_2(x)) \end{array}$$

such that

$$(*) \qquad \begin{cases} \alpha_1 \circ \beta_1 + \alpha_2 \circ \beta_2 = 1_{\mathbb{R}} \\ \beta_1 \circ \alpha_1 = 1_{\mathbb{R}} \, , \quad \beta_1 \circ \alpha_2 = 0 \\ \beta_2 \circ \alpha_1 = 0 \, , \quad \beta_2 \circ \alpha_2 = 1_{\mathbb{Q}} \end{cases}$$

We define a mapping $h : \mathbb{R} \to \mathrm{End}_{\mathbb{Q}}(\mathbb{R})$ by

$$h(y)(x) = \alpha_1(y\beta_1(x))$$

where $\mathbb{R} \oplus \mathbb{Q}$ is a direct sum of $\mathbb{R}$ and $\mathbb{Q}$. It is clear that $h(y)$ is a $\mathbb{Q}$-linear transformation for any $y \in \mathbb{R}$. By $(*)$, h is a ring morphism. If $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is constant, then $f(x, y) = 1$ as functions and $\hat{\hat{f}}(y) = 0$ as $\mathbb{Q}$-linear transformations. According to Theorem 9, we have a non-constant 2-variable function $\bar{\bar{h}} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ satisfying the conditions (Ex1), (Ex2) and (Ex3) such that $\bar{\bar{h}}(x, y)$ is not equal to x^y as functions.

□

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指数法則と環準同型写像

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数学分野

要　　旨

我々は，正の実数と実数の積集合から正の実数への指数法則を満たす２関数 $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+$ 全体の集合と実数$\mathbb{R}$から実数$\mathbb{R}$の$\mathbb{Q}$上線形変換全体への環準同型写像全体の集合の間に一対一の対応があることを示した。その応用として，関数x^yと異なる指数法則を満たす２関数$f(x, y)$が存在することを示した。

キーワード: 指数法則，環準同型写像

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