Dual modules of symmetric algebras over commutative rings

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Yoshimi KITAMURA*

Department of Mathematics

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Abstract

The purpose of this paper is to extend a recent result of W. Willems and A. Zimmermann on self-dual modules of group rings of finite groups over fields to symmetric algebras with anti-algebra automorphisms over commutative rings. They have proved the following. Let K be a field. Let $K[G]$ and $K[H]$ be group rings of finite groups G and H over K respectively. If M is a self-dual ($K[G]$, $K[H]$)-bimodule which is projective as a $K[G]$-module and V is a self-dual $K[G]$-left module, then $M \otimes_{K[G]} V$ is a self-dual $K[H]$-left module. It is well known that group rings of finite groups over fields are symmetric algebras with anti-algebra automorphisms. In this paper, generalizing fields to commutative rings, we consider algebras with anti-algebra automorphisms over commutative rings and obtain the following result. Let k be a commutative ring. Let R be an algebras over k with anti k-algebra endomorphism $\rho : R \to R$ and S be a symmetric algebras over k with anti k-algebra automorphism $\sigma : S \to S$. Let M and N be (R, S)-bimodules such that M is isomorphic to $\mathrm{Hom}_k(N, k)$ as an (R, S)-bimodule and N is finitely generated projective as a right S-module. Let V and W be left S-modules such that V is isomorphic to $\mathrm{Hom}_k(W, k)$ as a left S-module. Then $M \otimes_S V$ is isomorphic to $\mathrm{Hom}_k(N \otimes_S W, k)$ as a left R-module. For a field K and finite groups G, H, setting $k = K$, $N = M$, $W = V$, this result yields their result immediately.

Key words: bi-linear form, dual module, symmetric algebra, anti-homomorphism.

Mathematics Subject Classification (2010): Primary 15A63, 16D20; Secondary 16H99, 16W20.

Department of Mathematics, Tokyo Gakugei University, 4-1-1 Nukuikita-machi, Koganei-shi, Tokyo 184-8501, Japan
E-mail: kitamura@u-gakugei.ac.jp

* Tokyo Gakugei University (4-1-1 Nukui-kita-machi, Koganei-shi, Tokyo 184-8501, Japan)

1 Introduction

The aim of this note is to extend the result of W. Willems and A. Zimmermann [2, Proposition 3.3] to algebras over commutative rings.

They have considered group rings of finite groups over fields in there. Such group rings are symmetric algebras with anti-automorphisms. We shall consider algebras with anti-algebra endomorphisms over commutative rings in this note.

Let k be a commutative ring. First we deal with dual modules of k-modules, Frobenius algebras and symmetric algebras over k relative to bi-linear forms. Second we study two-sided modules and k-algebras with anti-algebra homomorphisms, in particular, symmetric k-algebras with anti-automorphisms. Let R and S be algebras over k with anti k-algebra homomorphisms $\rho : R \to R$ and $\sigma : S \to S$. Then, for every left R-module X, its dual module $\mathrm{Hom}_k(X, k)$ becomes a left R-module via ρ. Moreover, if M is an (R, S)-bimodule, then $\mathrm{Hom}_k(M, k)$ becomes an (R, S)-bimodule via ρ and σ. The result mentioned in the abstract above is our main one of the present paper. This extends Proposition 3.3 of [2] to symmetric algebras with anti-automorphisms over commutative rings.

2 Dual modules over commutative rings

Throughout this note, every ring is associative with identity, every module is unital and k denotes a commutative ring. Further, for a k-module X, the k-dual module $\mathrm{Hom}_k(X, k)$ of X is denoted by X^*. We refer to [1] for other unexplained terminology.

Let begin by recalling the following well-known lemma. For the reader's conveniences, we give the proof.

Lemma 2.1 *For k-modules M and N, the following are equivalent:*

(1) *$M \cong N^*$ as k-modules and N is finitely generated projective as a k-module.*

(2) *There exist a non-degenerate k-bilinear form $(-,-) : M \times N \to k$ and a finite number of elements $x_i \in M, y_i \in N$ $(i = 1,\ldots,n)$ such that $y = \sum_{i=1}^n (x_i, y)y_i$ for all y in N.*

Proof. (1)⇒(2). Let $\varphi : M \to N^*$ be a k-isomorphism. Define $(-,-) : M \times N \to k$ by $(x, y) = \varphi(x)(y)$ for $x \in M, y \in N$. Then $(-,-)$ is a k-bilinear form such that $(x, y) = 0$ for all $y \in Y$ implies $x = 0$. Since N is a finitely generated projective k-module, there exist a finite number of elements $g_i \in N^*, y_i \in N$ $(i = 1,\ldots,n)$ with $y = \sum_i g_i(y)y_i$ for all $y \in N$. For each $i = 1,\ldots,n$, there exists an element x_i of M with $g_i = \varphi(x_i)$. Then we have $y = \sum_{i=1}^n (x_i, y)y_i$. Moreover assume that $(x, y) = 0$ for all $x \in M$. Then we have $y = \sum_{i=1}^n (x_i, y)y_i = 0$. Hence the k-bilinear form $(-,-)$ is non-degenerate.

(2)⇒(1). Let $(-,-) : M \times N \to k$ be a k-bilinear form and $x_i \in M, y_i \in N$ $(i = 1,\ldots,n)$ elements satisfying the conditions in (2). Define $\varphi : M \to N^*$ by $\varphi(x)(y) = (x, y)$ for $x \in M, y \in N$. Then φ is a k-homomorphism evidently. Assume $\varphi(x) = 0$. Then, since $(x, y) = 0$ for all y in N, we have $x = 0$, and so φ is a monomorphism. For each $i = 1,\ldots,n$, let set $g_i = \varphi(x_i) \in N^*$. Then we have $\sum_i g_i(y)y_i = \sum_i (x_i, y)y_i = y$, and so N is a finitely generated projective k-module. Let g be a k-homomorphism of N to k. Setting $x = \sum_i g(y_i)x_i$, we have

$$\varphi(x)(y) = (x, y) = (\sum_i g(y_i)x_i, y) = \sum_i g(y_i)(x_i, y) = g(\sum_i (x_i, y)y_i) = g(y) \quad (y \in N),$$

and so $g = \varphi(x)$. It follows that φ is a k-isomorphism. □

Corollary 2.2 *For a k-module M, the following are equivalent:*

(1) *M is finitely generated projective as a k-module and $M \cong M^*$ as k-modules.*

(2) *There exist a non-degenerate k-bilinear form $(-,-) : M \times M \to k$ and a finite number of elements $x_i, y_i \in M$ $(i = 1,\ldots,n)$ such that $x = \sum_{i=1}^n (x_i, x)y_i = \sum_{i=1}^n (x, y_i)x_i$ for all x in M.*

Proof. (1)⇒(2). By Lemma 2.1, there exist a non-degenerate k-bilinear form $(-,-) : M \times M \to k$ and a finite number of elements x_i, y_i in M such that $x = \sum_i (x_i, x)y_i$ for all $x \in M$. Since

$$(\sum_i (x,y_i)x_i, y) = \sum_i (x,y_i)(x_i,y) = (x, \sum_i (x_i,y)y_i) = (x,y) \quad (\forall y \in M),$$

we have $x = \sum_i (x,y_i)x_i$. Thus we have (2). The implication (2)⇒(1) is evident from Lemma 2.1. □

Let R be an algebra over k. Let X be a left R-module. Then the k-dual module $X^*:= \mathrm{Hom}_k(X, k)$ becomes naturally a right R-module defined by

$$(f.r)(x) = f(rx) \quad (f \in X^*, r \in R, x \in X).$$

Similarly, for a right R-module Y, Y^* becomes a left R-module defined by

$$(r.g)(y) = g(yr) \quad (g \in Y^*, r \in R, y \in Y).$$

In particular $R^*:= \mathrm{Hom}_k(R, k)$ becomes an (R, R)-bimodule defined by

$$(a.f.b)(r) = f(bra) \quad (a,b,r \in R, f \in R^*).$$

Proposition 2.3 *For a k-algebra R, the following are equivalent:*

(1) *R is finitely generated projective as a k-module and $R \cong R^*$ as right R-modules.*

(2) *There exist a non-degenerate k-bilinear form $(-,-) : R \times R \to k$ and a finite number of elements $x_i, y_i \in R$ $(i = 1,\ldots,n)$ such that*

$$(xy,z) = (x,yz) \quad (x,y,z \in R),$$

$$x = \sum_i (x_i,x)y_i = \sum_i (x,y_i)x_i \quad (x \in R).$$

(3) *R is finitely generated projective as a k-module and $R \cong R^*$ as left R-modules.*

If the k-algebra R satisfies these equivalent conditions, then R is called a Frobenius k-algebra.

Proof. (1)⇒(2). Let $\varphi : R \to R^*$ be an isomorphism as a right R-module. As shown in the proof of Corollary 2.2, the k-bilinear form $(-,-) : R \times R \to k$ defined by $(x,y) = \varphi(x)(y)$ for x, y in R is non-degenerate and there exist a finite number of elements x_i, y_i $(i = 1,\ldots,n)$ in R such that

$$x = \sum_i (x_i,x)y_i = \sum_i (x,y_i)x_i \quad (x \in R).$$

Further we have $(xy,z) = \varphi(xy)(z) = (\varphi(x).y)(z) = \varphi(x)(yz) = (x,yz)$.

(2)⇒(1). Let $(-,-) : R \times R \to k$ be a k-bilinear form and x_i, y_i $(i = 1,\ldots,n)$ elements in R satisfying the conditions in (2). Define $\varphi : R \to R^*$ by $\varphi(x)(y) = (x,y)$ for x, y in R. By the proof of Lemma 2.1, φ is a k-isomorphism. Since $(xy, z) = (x, yz)$, we have $\varphi(xy)(z) = (\varphi(x).y)(z)$. Hence φ is a right R-homomorphism, and so, it is a right R-isomorphism. Noting $x = \sum_i (x_i,x)y_i$, R is a finitely generated projective k-module. Therefore we have (1).

We can show the equivalence (2)⇔(3) similarly. □

Remark 1 If R is a Frobenius algebra over k and x_i, y_i $(i = 1,\ldots,n)$ are elements of R given in Proposition 2.3 (2), then, for all $r \in R$, we have $\sum_i ry_i \otimes x_i = \sum_i y_i \otimes x_i r$ in $R \otimes_k R$.

This can be shown as follows. Let $\varphi : R \to R^*$ be the R-isomorphism given in (2)⇒(1). Since R is a finitely generated projective k-module, there exists an isomorphism $\theta : R \otimes_k R^* \to \mathrm{Hom}_k(R, R)$ given by $\theta(r \otimes f) = r.f$. Then the element $\sum_i y_i \otimes x_i$ of $R \otimes_k R$ is corresponding to the identity element of $\mathrm{Hom}_k(R, R)$ in the composite $R \otimes_k R \cong R \otimes_k R^* \cong \mathrm{Hom}_k(R, R)$. It follows that $\sum_i ry_i \otimes x_i = \sum_i y_i \otimes x_i r$ for all $r \in R$. □

Corollary 2.4 *For a k-algebra R, the following are equivalent:*

(1) *R is finitely generated projective as a k-module and $R \cong R^*$ as (R, R)-bimodules.*

(2) *There exist a non-degenerate k-bilinear form $(-,-) : R \times R \to k$ and a finite number of elements $x_i, y_i \in R$ $(i = 1,\ldots,n)$ such that*

$$(xy,z) = (x,yz), \quad (x,y) = (y,x) \quad (x,y,z \in R),$$

$$x = \sum_i (x_i,x)y_i = \sum_i (x,y_i)x_i \quad (x \in R).$$

If the k-algebra R satisfies these equivalent conditions, then R is called a symmetric k-algebra.

Proof. (1)⇒(2). Let $\varphi : R \to R^*$ be an isomorphism as an (R, R)-bimodule. Then we have $x.\varphi(1) = \varphi(x) = \varphi(1).x$ for x in R. For $x, y \in R$, let define $(x, y) \in k$ by $(x, y) = \varphi(x)(y)$. Then we have $(x, y) = (y, x)$ and $(xy, z) = (x, yz)$. Further, as shown above, the k-bilinear form $(-,-) : R \times R \to k$ is non-degenerate and there exist a finite number of elements $x_i, y_i \in R$ $(i = 1, \ldots, n)$ satisfying the condition in (2).

(2)⇒(1). Let $(-,-) : R \times R \to k$ be a k-bilinear form given in (2). Define $\varphi : R \to R^*$ by $\varphi(x)(y) = (x, y)$ for x, y in R. Then we have

$$\varphi(uxv)(z) = (uxv, z) = (u, xvz) = (xvz, u) = (x, vzu) = (u.\varphi(x).v)(z) \quad (u, v, x, z \in R).$$

Thus φ is an isomorphism as an (R, R)-bimodule by the proof of Proposition 2.3. □

3 Symmetric algebras with anti-algebra automorphisms

Let R be an algebra over k. Recall that an anti k-algebra homomorphism $\rho : R \to R$ is a k-module homomorphism with $\rho(ab) = \rho(b)\rho(a)$ for all $a, b \in R$ and $\rho(1) = 1$.

Henceforth R and S denote k-algebras with anti k-algebra homomorphisms $\rho : R \to R$ and $\sigma : S \to S$.

Let X be a left R-module. Then X becomes a right R-module defined by $x * r = \rho(r)x$ for $r \in R, x \in X$. This module is denoted by X_ρ. If Y is a right S-module, then the left S-module ${}_\sigma Y$ is similarly defined by $s * y = y\sigma(s)$ for $s \in S, y \in Y$. Further, if M is an (R, S)-bimodule, then the (S, R)-bimodule ${}_\sigma M_\rho$ is defined by $s * m * r = \rho(r)m\sigma(s)$ for $r \in R, s \in S, m \in M$.

For a left (resp. right) R-module X, X^* becomes a right (resp. left) R-module defined by $(f.r)(x) = f(rx)$ (resp. $(r.f)(x) = f(xr)$) for $r \in R, f \in X^*, x \in X$. Therefore the left (resp. right) R-module ${}_\rho X^*$ (resp. X^*_ρ) may be defined as mentioned above. Moreover if M is an (R, S)-bimodule, then we can consider the (R, S)-bimodule ${}_\rho M^*_\sigma$ naturally;

$$(r * f * s)(m) = f(\rho(r)m\sigma(s)) \quad (r \in R, s \in S, f \in M^*, m \in M).$$

It is clear that $X \cong Y$ as right R-modules implies ${}_\rho X \cong {}_\rho Y$ as left R-modules and that $M \cong N$ as (R, S)-bimodules does ${}_\sigma M_\rho \cong {}_\sigma N_\rho$ as (S, R)-bimodules.

Definition 1 Let M and N be (R, S)-bimodules. A k-bilinear form $(-,-) : M \times N \to k$ is said to be (R, S)-*invariant* provided that $(rxs, y) = (x, \rho(r)y\sigma(s))$ for all $r \in R, s \in S, x \in M, y \in N$.

Remark 2 Let G and H be finite groups. Let R and S be group rings of G and H over k respectively. Let $\rho : R \to R$ and $\sigma : S \to S$ be the mappings defined by

$$\rho(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g g^{-1} \quad (a_g \in k),$$

$$\sigma(\sum_{h \in H} b_h h) = \sum_{h \in H} b_h h^{-1} \quad (b_h \in k),$$

Then R and S are symmetric algebras over k with anti k-algebra isomorphisms ρ and σ such that $\rho^2 = I$ and $\sigma^2 = I$. It is easy to see that, for (R, S)-bimodules M and N, a k-bilinear form $(-,-) : M \times N \to k$ is (R, S)-invariant iff $(gxh, gyh) = (x, y)$ for all $x \in M, y \in N, g \in G, h \in H$.

Lemma 3.1 *Let M and N be (R, S)-bimodules. Suppose that N is finitely generated projective as a right S-module. Then the following are equivalent:*

(1) $M \cong {}_\rho N^*_\sigma$ *as (R, S)-bimodules.*

(2) *There exists a non-degenerate k-bilinear form $(-,-) : M \times N \to k$ such that $(-,-)$ is (R, S)-invariant.*

Proof. (1)⇒(2). Let $\varphi : M \to {}_\rho N^*_\sigma$ be an isomorphism as an (R, S)-bimodule. Define $(-,-) : M \times N \to k$ by $(x, y) = \varphi(x)(y)$ for $x \in M$ and $y \in N$. Then $(-,-)$ is a non-degenerate k-bilinear form by the proof of Lemma 2.1. Since φ is an (R, S)-homomorphism, we have $\varphi(rxs) = r * \varphi(x) * s$ for $r \in R, x \in X, s \in R$, and so,

$$(rxs, y) = \varphi(rxs)(y) = (r * \varphi(x) * s)(y) = \varphi(x)(\rho(r)y\sigma(s)) = (x, \rho(r)y\sigma(s)),$$

from which we have $(rxs, y) = (x, \rho(r)y\sigma(s))$. It follows that $(-,-)$ is (R, S)-invariant.

(2)⇒(1). Let $(-,-) : M \times N \to k$ be a k-bilinear form given in (2). Define $\varphi : M \to {}_\rho N^*_\sigma$ by $\varphi(x)(y) = (x, y)$ for $x \in M, y \in N$. Then φ is a k-isomorphism. Since the k-bilinear form $(-,-)$ is (R, S)-invariant, φ is an (R, S)-linear map. Hence we have $M \cong {}_\rho N^*_\sigma$ as (R, S)-bimodules. □

Let X be an (R, S)-bimodule. Then $\mathrm{Hom}(X_S, S_S)$ consisting of all S-homomorphisms of X to S becomes an (S, R)-bimodule given by

$$(s.\alpha.r)(x) = s\alpha(rx) \quad (r \in R, s \in S, x \in X, \alpha \in \mathrm{Hom}(X_S, S_S)).$$

Thus we have the (R, S)-bimodule ${}_\rho\mathrm{Hom}(X_S, S_S)_\sigma$ defined by $r * \alpha * s = \sigma(s).\alpha.\rho(r)$.

The following is well-known. For the reader's conveniences, we give the proof.

Lemma 3.2 *Let A and B be rings. Let X be a right A-module, Y an (A, B)-bimodule and Z a right B-module. If X is a finitely generated projective right A-module, then there exists a $\mathbb{Z}$-isomorphism*

$$\lambda: \mathrm{Hom}(Y_B, Z_B) \otimes_A \mathrm{Hom}(X_A, A_A) \to \mathrm{Hom}(X \otimes_A Y_B, Z_B)$$

given by

$$\lambda(g \otimes \alpha)(x \otimes y) = g(\alpha(x)y) \quad (g \in \mathrm{Hom}(Y_B, Z_B),\ \alpha \in \mathrm{Hom}(X_A, A_A),\ x \in X, y \in Y).$$

Proof. Since X is a finitely generated projective A-module, there exist a finite number of elements $\alpha_i \in \mathrm{Hom}(X_A, A_A)$ and $x_i \in X$ $(i = 1, \dots, n)$ such that $x = \sum_i x_i\alpha_i(x)$ for all $x \in X$. Define

$$\mu: \mathrm{Hom}(X \otimes_A Y_B, Z_B) \to \mathrm{Hom}(Y_B, Z_B) \otimes_A \mathrm{Hom}(X_A, A_A)$$

by $\mu(\gamma) = \sum_i \gamma_i \otimes \alpha_i$ for $\gamma \in \mathrm{Hom}(X \otimes_A Y_B, Z_B)$, where $\gamma_i \in \mathrm{Hom}(Y_B, Z_B)$ is given by $\gamma_i(y) = \gamma(x_i \otimes y)$ for $y \in Y$. Noting $x = \sum_i x_i\alpha_i(x)$, we have $\mu \circ \lambda = I$, $\lambda \circ \mu = I$. It follows that λ is an isomorphism. □

Lemma 3.3 *Suppose that S is a symmetric k-algebra. For an (R, S)-bimodule X, we have an (S, R)-bimodule isomorphism*

$$\mathrm{Hom}_k(X, k) \cong \mathrm{Hom}(X_S, S_S).$$

Proof. Noting that S is a symmetric algebra over k, let $(-,-) : S \times S \to k$ be a k-bilinear form and s_i, t_i $(i = 1, \dots, n)$ elements of S satisfying the conditions in Corollary 2.4 (2). Define $\varphi : S \to S^*$ by $\varphi(s)(t) = (s, t)$ for $s, t \in S$. Then φ is isomorphic as an (S, S)-bimodule. Setting $h = \varphi(1)$, we have $h(st) = h(ts)$ for all $s, t \in S$. As mentioned in Remark 1, we have $\sum_i st_i \otimes s_i = \sum_i t_i \otimes s_i s$ in $S \otimes_k S$, and so, $\sum_i \beta(xst_i)s_i = \sum_i \beta(xt_i)s_i s$, where $\beta \in \mathrm{Hom}_k(X, k)$, $s \in S, x \in X$. Hence we have the mapping Tr of $\mathrm{Hom}_k(X, k)$ to $\mathrm{Hom}(X_S, S_S)$ defined by

$$Tr(\beta)(x) = \sum_i \beta(xt_i)s_i \quad (\beta \in \mathrm{Hom}_k(X, k),\ x \in X).$$

On the other hand, we have the mapping θ of $\mathrm{Hom}(X_S, S_S)$ to $\mathrm{Hom}_k(X, k)$ defined by

$$\theta(\alpha) = h \circ \alpha \quad (\alpha \in \mathrm{Hom}(X_S, S_S)).$$

Then it is easy to see that $\theta \circ Tr = I, Tr \circ \theta = I$ and that θ is a right R-homomorphism. Since $h(st) = h(ts)$ for all $s, t \in S$, θ is also a left S-homomorphism. This completes the proof. □

Lemma 3.4 *Let P be a right S-module and Q an (S, R)-bimodule. Then the mapping ν of $Q_\sigma \otimes_S {}_\sigma P$ to $P \otimes_S Q$ given by $\nu(q \otimes p) = p \otimes q$ may be defined and it is a $\mathbb{Z}$-homomorphism. Further, $Q_\sigma \otimes_S {}_\sigma P$ has a structure of right R-module given by $(q \otimes p)r = qr \otimes p$ for $r \in R$. Moreover, if σ is an anti k-isomorphism, then ν is an isomorphism whose inverse is given by $p \otimes q \mapsto q \otimes p$.*

Proof. The first statement is clear. Suppose that σ is anti-isomorphic. Let $p \in P, q \in Q, s \in S$. Since $\sigma^{-1}(s) * p = ps, q * \sigma^{-1}(s) = sq$, we have $q \otimes ps = sq \otimes p$ in $Q_\sigma \otimes_S {}_\sigma P$. Hence the mapping ν' of $P \otimes_S Q$ to $Q_\sigma \otimes_S {}_\sigma P$ given by $\nu'(p \otimes q) = q \otimes p$ may be induced. It is evident that ν and ν' are mutually inverse isomorphisms. □

Proposition 3.5 *Suppose that S is a symmetric k-algebra and that σ is an anti k-isomorphism. Let M and N be (R, S)-bimodules such that $M \cong {}_\rho N^*_\sigma$ as (R, S)-bimodules and that N is finitely generated projective as a right S-module. Let V and W be left S-modules such that $V \cong {}_\sigma W^*$ as left S-modules. Then there holds $M \otimes_S V \cong {}_\rho((N \otimes_S W)^*)$ as left R-modules.*

Proof. Using the assumption, we have the following left R-module isomorphisms:

$$\begin{aligned} M \otimes_S V &\cong {}_\rho N^*_\sigma \otimes_S {}_\sigma W^* \\ &\cong {}_\rho \mathrm{Hom}(N_S, S_S)_\sigma \otimes_S {}_\sigma W^* \quad \text{(Lemma 3.3)} \\ &\cong {}_\rho(W^* \otimes_S \mathrm{Hom}(N_S, S_S)) \quad \text{(Lemma 3.4)} \\ &\cong {}_\rho((N \otimes_S W)^*) \quad \text{(Lemma 3.2).} \end{aligned}$$

This completes the proof. □

The following extends Proposition 3.3 of W. Willems and A. Zimmermann [2] slightly.

Corollary 3.6 *Suppose that S is a symmetric k-algebra and that σ is an anti k-isomorphism. Let M be an (R, S)-bimodule such that $M \cong {}_\rho M^*_\sigma$ as (R, S)-bimodules and that M is finitely generated projective as a right S-module. Let V be a left S-module such that $V \cong {}_\sigma V^*$ as left S-modules. Then there holds $M \otimes_S V \cong {}_\rho((M \otimes_S V)^*)$ as left R-modules.*

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可換環上の対称多元環の双対加群

北　村　　　好

数学分野

要　旨

本論文の目的は体上の有限群の群環の双対加群に関する W. Willems と A. Zimmermann による最近の結果を可換環上の多元環の双対加群に拡張することである．彼等は次の事柄を示した：Kを体とする．$K[G]$, $K[H]$をそれぞれ有限群G, HのK上の群環とする．Mが自己双対$(K[G], K[H])$−両側加群で$K[G]$−加群として射影的であり，Vが自己双対$K[G]$−左加群ならば，$M \otimes_{K[G]} V$も自己双対$K[H]$−左加群である．よく知られているように体上の有限群の群環は自己反同型をもつ対称多元環である．体を可換環に拡張して，可換環上の多元環で自己反同型をもつものを考え以下の事柄を示す：kを可換環とする．Rをk上の多元環でk−反準同型写像$\rho : R \to R$を有し，Sをk上の対称多元環でk−反同型写像$\sigma : S \to S$を有するものとする．M, Nを(R, S)−両側加群でMは(R, S)−加群として$\mathrm{Hom}_k(N, k)$に同型であり，Nは右S−加群として有限生成射影的であるとする．V, Wを左S−加群でVは左S−加群として$\mathrm{Hom}_k(W, k)$に同型であるとする．このとき，$M \otimes_S V$は左R−加群として$\mathrm{Hom}_k(N \otimes_S W, k)$に同型である．体$K$と有限群$G, H$に対して，上記彼等の結果は$k = K, R = K[G], S = K[H], N = M, W = V$についてこの結果を適用すれば直ちにしたがう．

キーワード: 双線型形式，双対加群，対称多元環，反準同型写像