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RINGS WITH TORSIONLESS INJECTIVE HULLS

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Abstract : We give a characterization of reflexive modules over QF-3' rings extending the concept of linearly compact modules.

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In [7] B. J. Müller proved that if R is a $P.$ $F.$ ring, i. e., R is self-injective and cogenerator on both sides, then a left (or right) R-module M is reflexive, if and only if M is linealy compact. On the other hand, genaralizing this result we proved in [6] that if R is a *QF-3* ring with a (unique) minimal faithful left module fR, where $f = f^2 \in R$, a left R-module M is reflexive, if and only if M is embedded into a direct product P of copies of the left R-module R such that P/M is torsionless and for any finitely solvable system of congrences $\{x \equiv fx_i \bmod M_i\}_{i \in I}$, where M_i is a submodule of M, is solvable. In this case RfR becomes a minimal dense left ideal of R and this ideal plays a important role for the characterization of reflexive modules.

Let us denote by $I({}_RM)$ the injective hull of the left module M over a ring R. We shall say that a ring R is *QF-3'*, If $I({}_RR)$ and $I(R_R)$ are torsionless. It is to be noted that recently, Gómez Pardo and Guil Asensio [1] gave a characterization of reflexive modules over QF-3' rings by a genelarized concept of linearly comapsctness introduced by Hoshino and Takashima [2]. Evidently, rings of the this type have no minimal dense left ideal in general. The purpose of this note is to study reflexive modules over *QF-3'* rings concerning certain two-sided ideals which are similar to minimal dense left ideals of *QF-3* rings.

In this paper a left R-module M is called *torsion free*, if M is embedded into a direct product of copies of $I({}_RR)$. A left ideal D of R is called *dense*, if the right annihilator of D in $I({}_RR)$ is zero. It is well known that a two-sided ideal of R is a dense left ideal, if and inly if it has no non-zero right annihilator in R. A submodule M of a left R-module N is said to be rationally closed, if N/M is torsion free. Let M be a torsion free left R-module. Set $Q(M)=\{m\in I({}_RM)$; there exists a dense left ideal D such that $Dm\subseteq M\}$. Especially, $Q(R)$ becomes an extension ring of R. In the following we denote this ring $Q(R)$ by Q. Then, Q is called a *maximal left quotient ring* of R (cf [3],[8]). In this case, $Q(M)$ becomes a left Q-module and it is called a quotient left module of M (with respect to Lambek torsion theory). From results of [4] it is easily seen that if R is QF-3', then a maximal left quotient ring coincides with a maximal right quotient ring. Throughout this paper every homomorphism between modules will be written on the opposite side of scalars.

Proposition 1. *Let R be a QF-3' ring with a maximal quotient ring Q. If T is the trace ideal of $I(R_R)$, then T is a dense left ideal of R, and TQ is contained in R.*

Proof. Suppose $0\neq r\in R$. Since $I(R_R)$ is torsionless, there exists $f\in \mathrm{Hom}\ (I(R_R)_R, R_R)$ such that $f(R)\neq 0$. Then, $f(1)\ r\neq 0$, where 1 is the identity element of R, implies T has no non-zero right annihilator in R.

Next, assume $q\in Q$, $f\in \mathrm{Hom}\ (I(R_R)_R, R_R)$ and $m\in I(R_R)$. As Q is a maximal right qutient ring of R, $I(R_R)$ becomes a right Q-module. Furthemore, every R-homomorphism from $I(R_R)$ to R is a Q-homomorphism, Therefore, we have $f(m)q = f(mq)\in R$ and hence $TQ\subseteq R$.

Lemma 2. *If a left module M over a QF-3' ring R is contained in $\Pi_{i\in I}R$, a direct product of copies of R. Then, the quotient module $Q(M)$ is contained in $\Pi_{i\in I}Q$. Furthermore, if M is embedded in $\Pi_{i\in I}R$ as a rationally closed submodule, then $TQ(M)$ is contained in M, where T is the trace ideal of $I(R_R)$.*

Proof. By Proposition 1 $T(\Pi_{i\in I}Q)\subseteq \Pi_{i\in I}R$. Since T is a dense left ideal of R, $\Pi_{i\in I}R$ is a dense submodule of $\Pi_{i\in I}Q$. Then, it is easy to see that $\Pi_{i\in I}Q$ is a quotient module of $\Pi_{i\in I}R$. Hence $Q(M)$ is contained in $\Pi_{i\in I}Q$. Further, assume M is rationally closed in $\Pi_{i\in I}R$. Then, we have $M = Q(M)\cap \Pi_{i\in I}R \supseteq TQ(M)$.

Let M be a left R-module. Put $M^*=\mathrm{Hom}({}_RM, {}_RR)$ and $M^{**}= \mathrm{Hom}(M^*_R, R_R)$. If M is isomorphic to M^{**}, canonically, M is said to be *reflexive*. If X is a subset of M^*, we shall put $r(X)=\{f\in M^*\ ;\ (X)f=0\}$, which is a right R-submodule of M.

Lemma 3. *Let R, T be the same as in Proposition 1 and M a reflexive left R-module. Assume $\{x\equiv m_i\ Mod\ M_i\}_{i\in I}$ is a finitely solvable system of congruences, where M_i is a rationally closed sumbmodule of M. Then, for every $t\in T$ the system of congruences $\{x\equiv tm_i\ Mod\ M_i\}_{i\in I}$ is solvable.*

Proof. Let $f\in \Sigma_{i\in I}r(M_i)$. Assume by finite subsets A and B of I we can write $f=\Sigma_{i\in A}f_i= \Sigma_{j\in B}f_j$, where $f_i\in r(M_i)$ and $f_j\in r(M_j)$. Put $C = A\cup B$. Since $\{x\equiv m_i\ Mod\ M_i\}_{i\in I}$ is finitely solvable, there exists $m_C \in M$ such that $(m_K) f_K = (m_C)f_K$, $k \in C$. Then, $\Sigma_{i \in A}(m_i)f_i = \Sigma_{i \in A}(m_C)f_i = \Sigma_{j \in B}(m_C)f_j = \Sigma_{j \in B}(m_j)f_j$. Hence we can define $\theta \in (\Sigma_{i \in A} r(M_i))^*$ by $\theta(f) = \Sigma_{i \in A}(m_i)f_i$. Assume $a \in I(R_R)$ and h is an R-homomorphism from $I(R_R)$ to R. There exists a canonical mapping $\bar{a} : R \to I(R_R)$. The R-homomorphism $\bar{a} \cdot \theta : \Sigma_{i \in A} r(M_i) \to I(R_R)$ is extended to an R-homomorphism $\theta' : M^* \to I(R_R)$. Since $h \cdot \theta'$ is an element of $M^{**} = M$, there exists $m \in$M such that $[h(a) \cdot \theta](f_i) = h \cdot \theta'(f_i) = (m)f_i = (h(a)m_i)f_i$ for every $f_i \in r(M_i)$. It follows that $(h(a)m_i - m)f_i = 0$ and then $h(a)m_i - m \in \{y \in M_i;\ (y)f_i = 0$ for every $f \in r(M_i)\} = M_i$, since M_i is rationally closed in M. Therefore, we can easily check that $\{x \equiv tm_i\ Mod\ M_i\}_{i \in I}$ is a solvable system of congruences for every $t \in T$. This completes the proof.

Assume M is a left module over a ring R. Then, from the same argument as in the proof of [6, Theorem 3] M is reflexive, if (and only if) M is embedded in a direct product of copies of R as a rationally closed submodule and M is embedded in M^{**} canonically as a dense submodule.

Thorem 4. *Let R be a QF-3' ring and M a left R-module. Then, the following conditions are equivalent ;*

(*i*) *M is reflexive.*

(*ii*) *M is embeded into a direct product P of copies of R and P/M is torsionless, and there exists a dense left ideal D satisfying ; ,*

(*a*) *If* $\{x \equiv m_i\ Mod\ M_i\}_{i \in I}$ *is a finitely solvable system of congruences, where* M_i *is a rationally closed submodule of M and* $m_i \in M$, *then the system of cogrunences* $\{x \equiv dm_i\ Mod\ M_i\}_{i \in I}$ *is solvable for every* $d \in D$.

(*b*) $DQ(K) \subseteq K$ *for every torsion free homomorphic image K of M.*

Proof. (*i*) $\Rightarrow$ (*ii*). Let T be the trace ideal of $I(R_R)$. Set $D = T^2$. Cleary, D is a dense left ideal by Proposition 1. In view of the proof of [6, Thorem 3] and Lemma 3 it suffices to show (b). Let Z be a rationally closed submodule of M. Since M/Z ($=K$, say) is a torsionless left R-module, it is embedded into K^{**}, canonically. From [5, Lemma 1.5] K^{**} is embedded into a direct product of copies of R as a rationally closed submodule and hence from Lemma 2 $TQ(K^{**}) \subseteq K^{**}$. If we show that TK^{**} is in K, we have $DQ(K) \subseteq K$ and this completes the proof. Let $\theta \in K^{**}$, $a \in I(R_R)$, $\bar{a} : R \to I(R_R)$ canonically and $h \in I(R_R)^*$. Let $\pi : K_R^* \to M_R^*$ be the canonical R-monomorphism. Since $\bar{a} \cdot \theta \in$Hom $(K_R^*, I(R_R)_R)$, there exists a mapping $\beta : M^* \to I(R_R)$ such that $\bar{a} \cdot \theta = \beta \cdot \pi$. As $h \cdot \beta \in M^{**}$ and M is reflexive, there exists $m \in M$ such that for every $f \in K^*$ $[h \cdot \beta](\pi(f)) = (m)[\pi(f)] = (m+Z)f$. On the other hand, $[h \cdot \beta](\pi(f)) = [h(a)\theta](f)$. Therefore, $h(a)\theta$ is realized by the element $m+Z$ of K and we have $TK^{**} \subseteq K$.

(*ii*)$\Rightarrow$(*i*). Let $\Psi \in M^{**}$ and $g_1, g_2 \cdots, g_n \in M^*$. Put $K = \{((x)g_1 \cdots, (x)g_n)\ ;\ x \in M\} \subseteq \oplus_{i=1}^{n} R$. By the proof of [6, Lemma 1] $(\Psi(g_1) \cdots, \Psi(g_n)) \in Q(K)$. As K is a torsion free homomorphic image of M, $D(\Psi(g_1), \cdots, \Psi(g_n)) \subseteq K$. Assume c, $d \in D$. For every $g \in M^*$ there exists $m_g \in M$ such that $c\Psi(g)=(m_g)g$. Then, we can see $\{x \equiv m_g \text{ Mod } Ker\ g\}_{g \in M^*}$ is a finitely solvable system and then $\{x \equiv dm_g \text{ Mod Ker g}\}_{g \in M^*}$ is solvable, i. e., there exists $m \in M$ such that $(m)g = (dm_g)g = dc\Psi(g)$ for every $g \in M^*$. It follows that $D^2\Psi$ is contained in M, and this implies M is a dence submodule of M^{**}. Then, the consequence is immediate.

If a ring R has a maximal two-sided quotient ring Q and M is a torsion free left R-module. Then $({}_RM)^*=\mathrm{Hom}({}_RM, {}_RR)$ is embedded in $({}_QQ(M))^*=\mathrm{Hom}({}_QQ(M), {}_QQ)$ canonically.

Theorem 5. *A left module M over a QF-3' ring R is reflexive, if and only if M is embedded in a direct product of copies of R as a rationally closed submodule and the left Q-module Q(M) is Q-reflexive.*

Proof. Let M be a torsioless left R-module. Assume $f \in ({}_QQ(M))^*$ and U the trace ideal of $I({}_RR)$. Then, by the left-right symmetry of Lemma 1 $(M)fU \subseteq R$ and this implies $({}_RM)^*$ is a dense R-submodule of the right R-module $({}_QQ(M))^*$. Similarly, $({}_RM)^{**}$ is a dense submodule of the right R-module $({}_QQ(M))^{**}$. Therefore, $({}_QQ(M))^{**}$ is the quotient module of $({}_RM)^{**}$. If M is reflexive, $({}_QQ(M))^{**}$. is the quoient module of M, i.e., $Q(M)=({}_QQ(M))^{**}$.

The "if" part is evident, since $M \subseteq ({}_RM)^{**} \subseteq ({}_QQ(M))^{**}=Q(M)$ and this implies M is a dense submodule M^{**}.

A ring R ia called left *QF-3*, if R has a (unique) minimal faithful left module, which is isomorphic to a direct summand every faithful left R-module. *QF-3* rings means left and right *QF-3*. As an application of Theorem 3 of [6] we show the following

Theorem 6. *Every reflexive left module over a QF-3 ring has finite Goldie dimension.*

Proof. Let R be a *QF-3* ring and M a reflexive left R-module. Suppose $A = \oplus_{i \in I} A_i$ is an infinite direct sum of submodules of M. Let $M_i = \oplus_{j \in I \setminus \{i\}} A_j$ and fR the minimal faithful right R-module, where f is an idempotent in R. We can take a non-zero element $m_i = fm_i \in M_i$ for every $i \in I$. Then, $\{x \equiv fm_i \text{ Mod } M_i\}_{i \in I}$ is finitely solvable and hence by [6, Theorem 3] there exists $m \in M$ such that $m-m_i \in M_i$, $i \in I$. Since $m \in A$, its i-component is $m_i \neq 0$ and this is a contaraction.

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