連続写像と線形順序空間

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Continuous maps and linearly ordered spaces

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Abstract

For spaces X and Y, let $f : X \to Y$ be a map, and $A \subset X$. Define $f|A : A \to Y$ by $(f|A)(x) = f(x)$, and $f^* : X \to f(X)$ by $f^*(x) = f(x)$.

Ordinarily, we assume that A is a subspace of X, and so is $f(X)$ of Y. Thus, if f is continuous, then so are $f|A$ and f^* On the other hand, for a subset A of a linearly ordered space (abbreviated LOTS) $(Z, \leq)$, the subspace topology (relative topology) on A need not coincide with the order topology induced by the restriction of $\leq$. For LOTS X and Y, we shall consider the continuity of f, $f|A$, or f^* in terms of the two topologies on $A \subset X$ or $f(x)$.

Keywords: continuous map, linearly ordered space, subspace, monotone map, homeomorphism, connected set

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1. Preliminaries

A couple $(X, \mathcal{T})$ (*or* $(X, \mathcal{T}_X)$) means a space with a topology $\mathcal{T}$, but we use the symbol X as a space if we need not specify $\mathcal{T}$ explicitly.

For $(X, \mathcal{T})$ and $A \subset X$, A is *open* (resp. *closed*) in X if $A \in \mathcal{T}$ (resp. $X \backslash A \in \mathcal{T}$). An open set in X containing $x \in X$ is denoted by a *nbd* $V(x)$.

A subset A of $(X, \mathcal{T})$ is a *subspace* of X if it has the *subspace topology* (*relative topology*, or *induced topology*) $A \cap \mathcal{T}$ $(= \{A \cap U \mid U \in \mathcal{T}\})$.

Let $X = (X, \leq)$ be a (linearly) ordered set (with at least two points). For $a, b \in X$ with $a < b$, let $(a, b) = \{x \in X \mid a < x < b\}$, $[a, b] = \{x \in X \mid a \leq x \leq b\}$, $(a, +\infty) = \{x \in X \mid a < x\}$, and $(-\infty, a) = \{x \in X \mid x < a\}$, etc.

Let $X = (X, \leq)$ be a space having the subbase $\{(a, +\infty), (-\infty, a) \mid a \in X\}$. Then X is called a *linearly ordered*

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(topological) space, abbreviated LOTS, and the topology on X called the *order topology* (or *interval topology*) induced by $\leq$. For these, see [2, 6] (or [3], etc.).

Every LOTS is a Hausdorff space; actually a (hereditarily) normal space (as is well-known).

The symbol $\mathbb{R}$ (resp. $\mathbb{Q}$; $\mathbb{Z}$) is the space of real numbers (rationals; integers) with the usual (order) topology.

Notation: A couple $(X, \leq)$ means a LOTS with the order topology induced by $\leq$, and the topology is denoted by $\mathcal{T}(\leq)$. For a subset A of $(X, \leq)$, we assume that A has the the order denoted by $\leq_A$, which is the restriction of $\leq$ to A.
The symbol $A \subset (X, \leq)$ means that A has the order topology $\mathcal{T}(\leq_A)$, unless otherwise stated.

Let A be a non-empty subset of $(X, \mathcal{T})$. Then A is *connected in* X if it can not be represented as the union of two non-empty disjoint sets in $A \cap \mathcal{T}$. A is *compact in* X if each cover of A by sets in $A \cap \mathcal{T}$ contains a finite subcover.

A non-empty subset A of $(X, \leq)$ is *convex* in X if for any $a, b \in A$ with $a \leq b$, $[a, b] \subset A$.

The following basic lemma is well-known ([2, 3, 6, 11], etc.).

Lemma 1. 1. *For $X = (X, \leq)$ and $A \subset X$, the following hold.*
(1) X is connected iff X has no jumps and no gaps (in the sense of Dedekind cut). Thus, (i) if A is connected in X, then A is convex in X, and (ii) if X is connected, then any convex set in X is connected in X.
(2) X is compact iff it has no gaps, and X has a minimal point and a maximal point.

For a subset A of $(X, \mathcal{T})$, A is *dense* in X if $V \cap A \neq \emptyset$ for any non-empty set $V \in \mathcal{T}$. For a subset A of $(X, \leq)$, let us call A *dense-order* in X (i.e., dense in X in the sense of order ([2])), if for each $x < y$ in X, there exists $z \in A$ with $x < z < y$. Every dense-order set in $(X, \leq)$ is dense in X.

For $A \subset (X, \leq)$, obviously $\mathcal{T}(\leq_A) \subset A \cap \mathcal{T}(\leq)$, but the two topologies on A need not be the same, even if A is closed and open in X; A is dense in X; or A is itself a connected, compact LOTS (cf. [9], or see Example 3.2(1) later). But, we have the following, as is well-known ([2, 3, 6, 11], etc.).

Lemma 1. 2. *Suppose that $A \subset (X, \leq)$ is connected; compact; convex; or dense-order in X. Then A is a subspace of X (equivalently, $A \cap \mathcal{T}(\leq) \subset \mathcal{T}(\leq_A)$).*

Notation: For sets X and Y, let $f : X \to Y$ be a map. For $A \subset X$, the symbol $f|A$ means a map from A into Y by $(f|A)(x) = f(x)$ (restriction of domain).
The symbol f^* means a map from X onto $f(X)$ by $f^*(x) = f(x)$ (restriction of range).

Let $X = (X, \mathcal{T}_X)$ and $Y = (Y, \mathcal{T}_Y)$. Let $f : X \to Y$ be a map. Then f is *continuous* if $f^{-1}(\mathcal{T}_Y) \subset \mathcal{T}_X$, here $f^{-1}(\mathcal{T}_Y) = \{f^{-1}(V) \mid V \in \mathcal{T}_Y\}$; equivalently, for each $x \in X$ and each nbd $W(f(x))$ in Y, there exists a nbd $V(x)$ in X with $f(V(x)) \subset W(f(x))$.
For $A \subset X$, and $f(X) \subset Y$, ordinarily we assume that A is a subspace of X, and $f(X)$ is a subspace of Y; see [1, 2], and other reference books. Then, for f being continuous, $f|A$ and f^* are continuous. However, for $X = (X, \leq)$ (resp. $Y = (Y, \leq)$), the subspace topology on the subset A of X (resp. $f(X)$ of Y) need not coincide with the order topology $\mathcal{T}(\leq_A)$ (resp. $\mathcal{T}(\leq_{f(X)})$).

In this paper, for a map $f : (X, \leq) \to (Y, \leq)$ with $A \subset X$, we assume that $A \subset (X, \leq)$ with the topology $\mathcal{T}(\leq_A)$, and

$f(X) \subset (Y, \leq)$ with the topology $\mathcal{T}(\leq_{f(X)})$, unless otherwise stated.

2. Results

The following is elementary, but assume that the subset A of X (resp. $f(X)$ of Y) has the topology $\mathcal{T}_A$ (resp. $\mathcal{T}_{f(X)}$), whch is not necessarily a subspace topology.

Lemma 2. 1. *For a map $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$, the following hold.*

(1) For $A \subset X$, let $\mathcal{T}_X \cap A \subset \mathcal{T}_A$. If f is continuous, then so is $f|A : A \to Y$.

In particular, for a subspace A of X, if f is continuous, then so is $f|A$.

(2) (i) For $\mathcal{T}_{f(X)} \subset f(X) \cap \mathcal{T}_Y$, if f is continuous, then so is $f^* : X \to f(X)$.

(ii) For $f(X) \cap \mathcal{T}_Y \subset \mathcal{T}_{f(X)}$, if f^* is continuous, then so is f.

In particular, for a subspace $f(X)$ of Y, f is continuous iff so is f^*.

Proof. For (1), since $f^{-1}(\mathcal{T}_Y) \subset \mathcal{T}_X$, $f^{-1}(\mathcal{T}_Y) \cap A \subset \mathcal{T}_A$ by $\mathcal{T}_X \cap A \subset \mathcal{T}_A$. Thus $f|A : A \to Y$ is continuous.In (2), for (i), for each $V \in \mathcal{T}_{f(X)}$, let $V = f(X) \cap V'$ for some $V' \in \mathcal{T}_Y$. Since f is continuous, $(f^*)^{-1}(V) = (f^*)^{-1}(f(X) \cap V') = f^{-1}(V') \in \mathcal{T}_X$. Then f^* is continuous. For (ii), for each $W \in \mathcal{T}_Y$, $f^{-1}(W) = (f^*)^{-1}(f(X) \cap W) = (f^*)^{-1}(W') \in \mathcal{T}_X$ for some $W' \in \mathcal{T}_{f(X)}$. Then f is continuous. □

For a set X, the map $1_X : X \to X$ denotes the *identity* map by $1_X(x) = x$. For sets X and Y with $X \subset Y$, the map $i_X : X \to Y$ denotes the *inclusion map* by $i_X(x) = x$. The following is obvious.

Remark 2. 2. (1) (i) The map $1_X : (X, \mathcal{T}_X) \to (X, \mathcal{T}'_X)$ is continuous iff $\mathcal{T}'_X \subset \mathcal{T}_X$.

(ii) The map $i_X : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous iff $\mathcal{T}_Y \cap X \subset \mathcal{T}_X$.

(2) (i) For a discrete space X, any $f : X \to Y$ and f^* are continuous.

(ii) For a non-discrete space X, make X to be a discrete space X^*. Then the map $1_{X^*} : X^* \to X$ is continuous, but the map $1_X : X \to X^*$ is not continuous.

For a space X, let $\mathcal{P}$ be a cover of X consisting of subspaces of X. Then X *is determined by* $\mathcal{P}$ ([4]) (or X *has the weak topology with respect to* $\mathcal{P}$ ([7])) if $U \subset X$ is open in X iff $U \cap P$ is relatively open in P for each $P \in \mathcal{P}$, here we can replace "open" by "closed".

Let us recall the following which is routinely shown.

Lemma 2. 3. *(1) Let X be a space determined by a cover $\mathcal{P}$. Then $f : X \to Y$ is continuous iff so is $f|P : P \to Y$ for each $P \in \mathcal{P}$. In particular, this remains true if $\mathcal{P}$ is an open cover (or a locally finite closed cover) consisting of subspaces of X.*

(2) $(X, \leq)$ has a (disjoint) open cover consisting of convex components $C(a) = \bigcup\{C \mid C$ is convex in X with $a \in C\}$ $(a \in X)$.

The following holds by Lemmas 1.2, 2.1(1), and 2.3.

Proposition 2. 4. *(1) For $A \subset (X, \leq)$, suppose that A is (*) connected; compact; convex; or dense-order in X. Then for a continuous map f from X into (a space) Y, $f|A$ is continuous.*

(2) Suppose that $(X, \leq)$ is determined by a cover $\mathcal{P}$ each of whose elements is (*) in (1). Then $f : X \to Y$ is continuous iff so is $f|P : P \to Y$ for each $P \in \mathcal{P}$. In particular, this remains true for an open cover $\mathcal{P}$ of X consisting of convex

components in X.

Theorem 2. 5. *(1) (a) For* $A \subset (X, \leq)$, A *is a subspace of* X *iff the map* $i_A : A \to X$ *is continuous.*

(b) For $A \subset (X, \leq)$*, the following are equivalent.*

(i) A *is a subspace of* X.

(ii) For any continuous map f *from* X *into any* $(Y, \leq)$, $f|A : A \to (Y, \leq)$ *is continuous.*

(2) Let f *be a continuous map from (a space)* X *into* $(Y, \leq)$*. Then* f^* *is continuous.*

Proof. In (1), for (a), the only if part holds by Remark 2.2(1). For the if part, $\mathcal{T}_X \cap A \subset \mathcal{T}_A$, here $\mathcal{T}_A = \mathcal{T}(\leq_A)$. Since $\mathcal{T}_A \subset \mathcal{T}_X \cap A$, $\mathcal{T}_A = \mathcal{T}_X \cap A$. Thus A is a subspace of X. For (b), (i) implies (ii) by Lemma 2.1(1). To see (ii) $\Rightarrow$ (i), for the continuous map $1_X : (X, \leq) \to (X, \leq)$, the map $i_A : A \to (X, \leq)$ is continuous by (ii). Then (i) holds by (a).

For (2), $\mathcal{T}_{f(X)} \subset f(X) \cap \mathcal{T}_Y$, here $\mathcal{T}_{f(X)} = \mathcal{T}(\leq_{f(X)})$. Then (2) holds by Lemma 2.1(2). □

The following holds by Theorem 2.5(2) with Proposition 2.4(1).

Proposition 2. 6. *Let* f *be a continuous map from (a space)* X *into* $(Y, \leq)$*. If* A *is a subspace of* X*, then* $g = f|A : A \to Y$ *is continuous, thus so is* $g^* : A \to f(A) \subset Y$*. When* $A \subset X = (X, \leq)$ *is connected; compact; convex; or dense-order in* X, g *and* g^* *are continuous.*

A surjection $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is *quotient* if $\mathcal{T}_Y = \{V \subset Y | f^{-1}(V) \in \mathcal{T}_X\}$. Quotient maps are continuous. Every quotient injection is a homeomorphism.

Theorem 2. 7. *For a map* f *from (a space)* X *into* $(Y, \leq)$*, let* f^* *be quotient. Then* f *is continuous iff* $f(X)$ *is a subspace of* Y.

Proof. The if part holds by Lemma 2.1(2), for f^* is continuous. For the only if part, we show $f(X) \cap \mathcal{T}_Y \subset \mathcal{T}_{f(X)}$, here $\mathcal{T}_{f(X)} = \mathcal{T}(\leq_{f(X)})$. Let $O' = f(X) \cap O \in f(X) \cap \mathcal{T}_Y$, $O \in \mathcal{T}_Y$. Then $(f^*)^{-1}(O') = (f^*)^{-1}(f(X) \cap O) = f^{-1}(O) \in \mathcal{T}_X$. Since f^* is quotient, $O' \in \mathcal{T}_{f(X)}$. Thus, $f(X) \cap \mathcal{T}_Y \subset \mathcal{T}_{f(X)}$. While, $\mathcal{T}_{f(X)} \subset f(X) \cap \mathcal{T}_Y$. Then $\mathcal{T}_{f(X)} = f(X) \cap \mathcal{T}_Y$, which shows $f(X)$ is a subspace of Y. □

A map f from a space X into a space Y is *open* (resp. *closed*) if for any open (resp. closed) set A in X, $f(A)$ is open (resp. closed) in Y. Every open and closed map need not be continuous (by the map 1_X in Remark 2.2(2)).

Remark 2. 8. (1) Every projection p from the product space $X \times Y$ onto X by $p(x, y) = y$ is a continuous, open map.

(2) Every continuous map f from a compact space X into a Hausdorff space Y is closed (indeed, each closed set F in X is compact in X, then $f(F)$ is compact in Y, hence closed in Y).

Corollary 2. 9. *For a map* f *from (a space)* X *into* $(Y, \leq)$*, let* f^* *be open or closed. Then* f *is continuous iff* f^* *is continuous, and* $f(X)$ *is a subspace of* Y*. When* X *is connected or compact, this remains true.*

Proof. The if part holds by Lemma 2.1(2). For the only if part, f^* is continuous (by Theorem 2.5(2)). But, since f^* is open or closed, f^* is quotient. Hence, $f(X)$ is a subspace of Y by Theorem 2.7. For the only if part of the latter part, note $f(X)$ is connected or compact in Y, thus $f(X)$ is a subspace of Y by Lemma 1.2. □

Let us say that a map $f : (X, \leq) \to (Y, \leq)$ is *monotone* if f is monotonically increasing (i.e., $x \leq y$ implies $f(x) \leq f(y)$), or monotonically decreasing (i.e, $x \leq y$ implies $f(y) \leq f(x)$). We call f *order-preserving* if it is monotonically increasing.

Lemma 2.10. *Every monotone bijection* $f:(X,\leq)\to(Y,\leq)$ *is a homeomorphism.*

Proof. Since f is a monotone bijection, for the subbases $\mathcal{B}$ for $(X,\leq)$, and $\mathcal{B}'$ for $(Y,\leq)$, routinely $f(\mathcal{B})=\mathcal{B}'$, which implies f is a homeomorphism. □

Lemma 2.11. *Let* $f:(X,\leq)\to(Y,\leq)$ *be order-preserving (resp. monotone) on a dense subset* D *of* X. *If* f^* *is continuous,* f *is order-preserving (resp. monotone).*

Proof. Suppose f is not order-preserving. Then there exist $a,b\in X$ with $a<b$, but $f(b)\prec f(a)$. Since f^* is continuous, there exist nbds $V(a)$, $V(b)$ in X such that for $x\in V(a)$ and $y\in V(b)$, $x<y$, but $f(y)\prec f(x)$. Since D is dense in X, there exist $x,y\in D$ with $x\in V(a)$, $y\in V(b)$, then $x<y$, so $f(x)\leq f(y)$. But $f(y)\prec f(x)$, a contradiction. The parenthetic part is similarly shown. □

A map $f:X\to Y$ is a *homeomorphic embedding* if f^* is a homeomorphism and $f(X)$ is a subspace of Y.

Corollary 2.12. *Let* $f:(X,\leq)\to(Y,\leq)$. *If* f^* *is a homeomorphism, (i), (ii), and (iii) below are equivalent. If* f^* *is a bijection which is monotone on a dense subset* D *of* X, *(i), (ii) are equivalent (when* $X=D$, *(i), (ii), (iii) are equivalent).*

(i) f *is continuous.*

(ii) f *is a homeomorphic embedding.*

(iii) $f(X)$ *is a subspace of* Y.

Proof. (i) $\Rightarrow$ (iii) holds by Theorem 2.7. (iii) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (i) holds by Lemma 2.1(2). For the latter part, note that (i) implies that f^* is monotone by Lemma 2.11, thus f^* is a homeomorphism by Lemma 2.10. □

Proposition 2.13. *Let* $f:(X,\leq)\to(Y,\leq)$ *be a continuous map, and* X *be connected. Then the following are equivalent.*

(i) f *is monotone.*

(ii) Every $(f^*)^{-1}(y)$ *is connected in* X.

(iii) Every $(f^*)^{-1}(y)$ *is convex in* X.

Proof. For (i) $\Rightarrow$ (iii), suppose some $f^{-1}(y)$ is not convex in X. Then there exist $a,b,c\in X$ such that $a<c<b$ with $a,b\in f^{-1}(y)$, but $c\notin f^{-1}(y)$. Thus $f(a)=f(b)=y$, but $f(c)\neq y$. This shows that f is not monotone.

(iii) $\Rightarrow$ (ii) holds by Lemma 1.1(1).

For (ii) $\Rightarrow$ (i), first the following holds by means of Lemma 1.1, noting X is connected.

(*) For any convex set $[a,b]$ in X, (*) $I=f([a,b])$ is connected, compact in Y; hence, I is a convex set in Y having max I and min I (actually, $[a,b]$ is connected, compact in X. Then I is connected, compact).

Now, suppose f is not monotone on some $[a,b]$ in X. Thus, using (*), we can assume that (i) max $f([a,b])=p=f(c)$ with $a<c<b$, $f(a),f(b)\neq p$; or (ii) min $f([a,b])=p'=f(c')$ with $a<c'<b$, $f(a),f(b)\neq p'$ (if f is not monotone on some $[a',b']$ in X, but $f([a',b'])$ has max $f(a')$ and min $f(b')$ for example. Then we can take $[a,b]\subset[a',b']$ satisfying (i) or (ii), for f is not monotone on $[a',b']$). We may assume (i). Let $I=f([a,b])$, and $C=I\backslash\{p\}\neq\emptyset$. Then C is connected in I, because C is convex in I, noting $p=$max I. Let $g=f|[a,b]:[a,b]\to I$. Since f is continuous, g is continuous by Proposition 2.6. Since $[a,b]$ is compact in X, g is a closed map (by Remark 2.8(2)). Hence g is a quotient map. Also, for any $y\in I$, $g^{-1}(y)\,(=f^{-1}(y)\cap[a,b])$ is convex in $[a,b]$, hence connected in $[a,b]$ (by Lemma 1. 1). Thus, $g^{-1}(C)$ is connected in $[a,b]$ by [1, VI.3.4] (or [2, Theorem 6.1.29]). But, $g^{-1}(C)=[a,b]\backslash f^{-1}(p)\;(\ni a,b)$ is

not connected in $[a, b]$. This is a contradiction. Hence, f is monotone. □

Remark 2.14. We have the following in view of [1.1, Proposition 2.3].

(1) Let $f : (X, \leq) \to Y$ be an open map. Suppose $A = f^{-1}(y)$ is connected in X with $|A| \geq 2$(or A contains a connected set C in X with $|C| \geq 2$). Then y is isolated in Y. If f is continuous with $\{y\}$ closed in Y, $X (\neq A)$ is not connected. (Indeed, A contains a non-empty open set (a, b) in X by Lemma 1.1(1), then $y = f((a, b))$ is isolated in Y. If f is continuous, A is open and closed in X, thus X is not connected).

(2) (a) For $X = (X, \leq)$ and $Y = (Y, \leq')$, $X \times Y$ is a LOTS by the lexicographic order $\preceq$ defined by $(x_1, y_1) \preceq (x_2, y_2)$ if $x_1 < x_2$, or $x_1 = x_2$ with $y_1 \leq' y_2$.

(b) If $X \times Y$ is the *product space*, then it is not a LOTS by any order if X contains a connected set C with $|C| \geq 2$, and Y is not discrete (by (1) with Remark 2.8(1)).

Corollary 2.15. *A continuous map $f : \mathbb{R} \to \mathbb{R}$ is monotone iff every $(f^*)^{-1}(y)$ is connected in X* ([2,6.1.H]).

Corollary 2.16. *Every homeomorphism $f : (X, \leq) \to (Y, \leq)$ with X connected is monotone* ([10]).

Theorem 2.17. *Let $f : (X, \leq) \to (Y, \leq)$, and X be connected. Then (i), (ii) below are equivalent. When f is an injection, (i), (ii), and (iii) are equivalent.*

(i) f is continuous.

(ii) f^* is continuous, and $f(X)$ is a subspace of Y.

(iii) f is a homeomorphic embedding.

Proof. The equivalence (i) between (ii) holds by Corollary 2.9. When f is an injection, for (ii) $\Rightarrow$ (iii), f^* is a monotone bijection by Proposition 2.13. Thus, f^* is a homeomorphism by Lemma 2.10. Hence (ii), (iii) are equivalent. □

Corollary 2.18. *For $f : \mathbb{R} \to (Y, \leq)$, the result in Theorem 2.17 holds.*

3. Examples

We give examples which are referred to in earlier parts of this paper.

Lemma 3.1. *For any infinite LOTS $X = (X, \leq)$, we can make X to be a discrete LOTS X^* as follows: Let $Y = X \times \mathbb{Z}$ be a discrete LOTS by the lexicographic order $\preceq$ (in Remark 2.14(2)(a)). Then there exists a bijection $f : X \to Y$ since $|X| = |Y|$, thus we can define the order $\leq_f$ on X by $x \leq_f y$ iff $f(x) \preceq f(y)$ on Y. Hence $f : (X, \leq_f) \to (Y, \preceq)$ is an order-preserving homeomorphism by Lemma 2.10. Then $X^* = (X, \leq_f)$ is a discrete LOTS.*

Related to Proposition 2.4, Theorem 2.5, and Corollary 2.12, etc., we have the following example. (In Lemma 1.2, note that every connected, compact LOTS need not be a subspace in $\mathbb{R}$ in view of Example 3.2(1) below).

Example 3.2. (1) A map $f = 1_X : (X, \leq) \to (X, \preceq)$ is a homeomorphism, but $f|A$ with $A \subset (X, \leq)$ is not continuous, where (i) $(X, \leq)$ and A are connected, compact LOTS, or (ii) $(X, \leq)$ is a discrete LOTS, and A is a connected, compact LOTS.

(2) A map $f = i_X : X \to (Y, \leq)$ with $X \subset (Y, \leq)$ is continuous, but the map $f^* = 1_X : X \to (Z, \preceq)$ is not continuous, where Z is a subset of Y, but $\preceq$ is not the restriction of $\leq$.

(3) A surjection (resp. bijection) $f : (X, \leq) \to (X, \leq)$ such that X is a connected, compact LOTS which is disjoint union of connected sets (resp. dense-ordered sets) A and B in X, and $f|A$, $f|B$ are order-preserving continuous, but f is neither

monotone nor continuous.

Proof. In (1), for (i), let $(X, \leq) = [0, 3] \subset \mathbb{R}$, and $A = [0, 1) \cup [2, 3] \subset (X, \leq)$. Then X and A are connected, compact LOTS by Lemma 1.1. For (ii), let $(X, \leq) = Y = (\mathbb{R} \times \mathbb{Z}, \leq)$, and let $A = [0, 1] \times 0 \subset Y$ in Lemma 3.1. Since A is homeomorphic to $[0, 1] \subset \mathbb{R}$, A is a connected, compact LOTS. Thus, in (i) and (ii), $f = 1_X : (X, \leq) \to (X, \leq)$ is a homeomorphism, but $f|A = i_A : A \to (X, \leq)$ is not continuous.

For (2), let $X = \mathbb{Q}$ in $Y = \mathbb{R}$, and let $Z = \mathbb{Q}^*$ (in Lemma 3.1). Then $f = i_X : X \to Y$ is continuous, but $f^* = 1_X : X \to Z$ is not continuous.

For (3), let $X = [0, 2] \subset \mathbb{R}$, and let $A = [0, 1)$, $B = [1, 2]$. Let $f : (X, \leq) \to (X, \leq)$ by $f(x) = x\ (x \in A)$, $f(x) = 2x - 2\ (x \in B)$. For the parenthetic part, let $X = [0, 2] \subset \mathbb{R}$, and let $A = [0, 2] \backslash \mathbb{Q}$, $B = [0, 2] \cap \mathbb{Q}$. Let $f : (X, \leq) \to (X, \leq)$ by $f(x) = x\ (x \in A)$, and $f(x) = (1/2)x\ (0 \leq x < 4/3, x \in B)$, $f(x) = 2x - 2\ (4/3 \leq x \leq 2, x \in B)$. □

Related to Theorem 2.5, Theorem 2.7, etc., we have the following example.

Example 3. 3. (1) An order-preserving map $f : (X, \leq) \to (Y, \leq)$ such that X and $f(X)$ are connected, compact LOTS, and f^* is a homeomorphism, but f is not continuous, and $f(X)$ is not a subspace of Y.

(2) An order-preserving map $f : (X, \leq) \to (Y, \leq)$ such that Y, $f(X)$ are connected LOTS, and f, f^* are continuous, but f^* is not quotient, and $f(X)$ is not a subspace of Y.

Proof. For (1), let $X = [0, 2], Y = [0, 3] \subset \mathbb{R}$, and $f : X \to Y$ by $f(x) = x\ (x \in [0, 1)), f(x) = x + 1\ (x \in [1, 2])$. Then X, and $f(X) = [0, 1) \cup [2, 3] \subset Y$ are connected, compact LOTS, but $f(X)$ is not a subspace of Y. f is an order-preserving injection, thus f^* is a homeomorphism (by Lemma 2.10), but f is not continuous.

For (2), let $X = [0, 1) \cup (1, \infty) \subset \mathbb{R}$ and $Y = \mathbb{R}$. Let $f(x) = 0\ (0 \leq x < 1), f(x) = x\ (1 < x)$. Here, $f(X) = \{0\} \cup (1, \infty)$ is a connected LOTS which is not a subspace of Y, and f is continuous. Also, f^* is continuous, but it is not quotient (indeed, $(f^*)^{-1}(0) = [0, 1)$ is open in X, but $\{0\}$ is not open in $f(X)$). □

Related to Proposition 2.13, Theorem 2.17, etc., we have the following example.

Example 3. 4. (1) (a) A homeomorphism $f : (X, \leq) \to (X, \leq)$ is not monotone.

(b) A bijection $f = 1_X : (X, \leq) \to (X, \leq)$ is continuous, but f is not a homeomorphism (not even quotient), and not monotone.

(2) An order-preserving continuous surjection $f : (X, \leq) \to (X, \leq)$, but some $f^{-1}(y)$ is not connected in X.

(3) A bijection $f : (X, \leq) \to (X, \leq)$ with X connected is not continuous.

(4) An order-preserving surjection $f : (X, \leq) \to (Y, \leq)$ such that X and every $f^{-1}(y)$ are connected, but f is not continuous.

Proof. In (1), for (a), let $X = \mathbb{R} \backslash \{0\}$, and let $f(x) = x\ (x < 0)$, $f(x) = 1/x\ (x > 0)$. For (b), let $(X, \leq) = \mathbb{R}^*$ in Lemma 3.1, and let $(X, \preceq) = \mathbb{R}$.

For (2), let $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x\ (x \leq 0)$, $f(x) = x - 1\ (x \geq 1)$.

For (3), let $X = \mathbb{R}$, and let $f(x) = x\ (x \leq 0)$, $f(x) = 1/x\ (x > 0)$.

For (4), let $X = \mathbb{R}$, and $Y = (-\infty, 0] \cup [1, \infty) \subset \mathbb{R}$. Let $f(x) = x\ (x \leq 0), f(x) = 1\ (0 < x \leq 1)$, and $f(x) = x\ (1 < x)$. □

We conclude this paper by recording some related matters around LOTS.

Note: As a case of LOTS, in [9] we consider algebraic order topologies on ordered groups or ordered rings, which are compatible with their operations (cf.[8]). In a separated paper, we will consider continuity of homomorphisms between ordered fields or ordered rings, etc.

Note: As generalizations of LOTS, let us recall the following spaces.

A space $(X, \mathcal{T})$ is *orderable* ([11] (or [6])) if $\mathcal{T}$ coincides with an order topology by some order on X. Every orderable space need not be a LOTS, and every subspace of a LOTS need not be orderable ([9, 11], etc.). A space X with an order is a *generalized ordered space* (abbreviated GO-space) if X is a subspace (or closed subspace) of a LOTS X', where the order of X is the restriction of the order of X'. Every GO-space is a LOTS if it is connected or compact. For GO-spaces, see [5, 6] etc. Every orderable space is a GO-space, but the converse need not hold. (We do not deal with these spaces in this paper, but we will leave it to the readers).

REFERENCES

[1] J. Dugundji, Topology, Allyn and Bacon, Inc., Boston, 1967.

[2] R. Engelking, General Topology, Heldermann Verlag Berlin, 1989.

[3] L. Gillman and M. Jerison, Rings of Continuous Functions, Van Nostrand Reinhold company, 1960.

[4] G. Gruenhage, E. Michael and Y. Tanaka, Spaces determined by point-countable covers, Pacific J. Math., 113(1984), 303-332.

[5] D. J. Lutzer, On generalized ordered spaces, Dissertationes Math., 89(1971), 1-32.

[6] J. Nagata, Modern General Topology, Elsevier Science Publishers B.V., North-Holland, 1985.

[7] Y. Tanaka, Point-countable k-systems and product of k-spaces, Pacific J. Math., 113(1982), 199-208.

[8] Y. Tanaka, Topology on ordered fields, Comment. Math. Univ. Carolinae, 53(2012), 139-147.

[9] Y. Tanaka and Y. Kitamura, Algebraic order topologies, and related matters, to appear in Tsukuba J. Math.

[10] Y. Tanaka and T. Shinoda, Orderability of compactifications, Questions and Answers in General Topology, 21(2003), 79-89.

[11] M. Venkataraman, M. Rajagopalan and T. Soundararajan, Orderable topological spaces, General Topology and Appl., 2(1972), 1-10.

連続写像と線形順序空間

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要　旨

f を空間 X から空間 Y への写像とし，$A \subset X$ とする。$f|A$ を A から Y への写像で $(f|A)(x) = f(x)$，f^* を X から $f(X)$への写像で $f^*(x) = f(x)$ とする。

通常，A，$f(X)$ をそれぞれ，X，Y の部分空間として考え，f が連続ならば，$f|A$，f^* は連続になる。一方，線形順序空間 $(Z, \leq)$ の部分集合 A において，部分空間位相(相対位相)は，$\leq$ から誘導された順序位相と必ずしも一致しない。線形順序空間 X，Y に対し，$A \subset X$ または $f(X) \subset Y$ における 2 つの位相の観点から，f，$f|A$，または f^* の連続性を考察する。

キーワード：連続写像，線形順序空間，部分空間，単調写像，位相写像，連結集合

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